A NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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Pettis[1] raised the question whether or not separability of the range space implies almost everywhere weak differentiability of Pettis integrals. Phillips[2] has given an example which answers this question in the negative. His construction is based on a sequence of orthogonal vectors in Hilbert space. We present here a different example of the same type of function. Our basic construction is that of a function defined to the space C. Using that function as a basis, we are able to give a specific construction of such a function defined to each member of a large class of Banach spaces.

1. Metric density properties of a non-dense perfect set. Let $B \subset [0, 1]$ be a non-dense perfect set of measure one-half, and let $\overline{B}$ be its complement. $\overline{B}$ may be constructed by taking the sum of a set of open intervals classified as follows:

1 interval of length $1/4$,
2 intervals each of length $1/16$,
4 intervals each of length $1/64$,
. . . .
2^{n-1} intervals each of length $1/2^{2n}$,
. . . .

We shall refer to the intervals of length $1/2^{2n}$ as *intervals of* $\overline{B}$ *of order* n. We shall assume that each interval of $\overline{B}$ of order n is the center portion of the space either between two intervals of $\overline{B}$ of lower order or between one such interval of $\overline{B}$ and an end point of the unit interval. These spaces we shall refer to as *gaps of order* n, and we shall denote such a gap by the symbol G_n. If $\overline{B}$ is constructed as noted above, then for each n, any two sets each of the form $G_n \cdot \overline{B}$ are congruent; hence we shall use G_n to denote a gap of order n, and we shall not find it necessary to specify which one.

The following three lemmas are now obvious.

1.1. LEMMA. $|\overline{B} \cdot G_n| = 1/2^{2n-1}$.

1.2. LEMMA. $|G_n| = 1/2^n + 1/2^{2n-1}$.

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[1] See [3, p. 303]. Numbers in brackets refer to the references cited at the end of the paper.

[2] See [4, p. 144].

1.3. DEFINITION. If I is any subinterval of $[0, 1]$, we define

$$\rho(I) = |\overline{B}\cdot I|/|I|.$$

1.4. LEMMA. $\rho(G_n)=1/(1+2^{n-1})$.

The following important lemma demonstrates a lower bound on the density function ρ. This lower bound may tend to zero as $|I|$ tends to zero, but it is independent of the location of I.

1.5. LEMMA. *If*

$$|I| \geqq |G_n| = 1/2^n + 1/2^{2n-1},$$

then

$$\rho(I) > 1/(2 + 2^{n-1}).$$

First suppose $|I| = |G_n|$. For all such I the G_n have minimum values for ρ; for if $I=G_n$ and then is moved a little to the right or left, some points of B are excluded and only points of $\overline{B}$ are included—thus obviously increasing $\rho(I)$—until the interval of $\overline{B}$ is covered. However, this interval of $\overline{B}$ is of order at most $n-1$ and hence has measure at least $1/2^{2n-2}$; thus so long as I contains this interval of $\overline{B}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^n + 1/2^{2n-1}} = \frac{2}{1 + 2^{n-1}} = 2\rho(G_n).$$

If I is moved on beyond this interval of $\overline{B}$ the above argument applies again by considering the movement in the reverse direction. Comparison with Lemma 1.4 now shows that Lemma 1.5 is established in case $|I| = |G_n|$.

Consider now the effect of increasing $|I|$. If $|I| = |G_n|$ and I contains no interval of $\overline{B}$ of order less than or equal to $n-1$, then one end point of I (let us assume it is the right-hand one) lies in the closure of such an interval of $\overline{B}$. Thus any small extension of I to the right will (until the interval of $\overline{B}$ is covered) add to I only points of $\overline{B}$, thus obviously increasing $\rho(I)$. If $I=G_n$ at the start, a small extension to the left will have the same effect. Otherwise an extension to the left may be regarded as a translation to the left and a subsequent extension to the right, and these cases have already been discussed. In case I contains an interval of $\overline{B}$ of order not greater than $n-1$, we have $|\overline{B}\cdot I| > 1/2^{2n-2}$; thus if $|I| \leqq |G_{n-1}| = 1/2^{n-1}+1/2^{2n-3}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^{n-1} + 1/2^{2n-3}} = \frac{1}{2 + 2^{n-1}}\,.$$

If $|I| > |G_{n-1}|$, the above argument may be repeated with $n-1$ substituted for n, thereby obtaining an even larger lower bound for $\rho(I)$. This proves Lemma 1.5 for all cases.

1.6. LEMMA. *For all* $I \subset [0, 1]$,

$$\rho(I) > |\overline{B} \cdot I|^{1/2}/2^{5/2}.$$

We shall prove Lemma 1.6 by showing that, for each n, the required inequality holds for

$$1/2^{2n+1} < |\overline{B} \cdot I| \leqq 1/2^{2n-1}.$$

This will cover all possibilities. For $|\overline{B} \cdot I|$ in this range, we consider first the case $|I| < |G_n| = 1/2^n + 1/2^{2n-1}$. In this case

$$\frac{1}{\rho(I)} = \frac{|I|}{|\overline{B} \cdot I|} < \frac{|I|}{1/2^{2n+1}} < \frac{1/2^n + 1/2^{2n-1}}{1/2^{2n+1}} = 4 + 2^{n+1}$$
$$= 4 + 2^{3/2}[1/2^{2n-1}]^{-1/2} \leqq 4 + 2^{3/2}|\overline{B} \cdot I|^{-1/2}.$$

Considering now the case $|I| \geqq |G_n|$ (and assuming $|\overline{B} \cdot I|$ still in the same range) we have, using Lemma 1.5 and the above inequalities,

$$1/\rho(I) < 2 + 2^{n-1} < 4 + 2^{n+1} \leqq 4 + 2^{3/2}|\overline{B} \cdot I|^{-1/2}.$$

Now for all $I \subset [0, 1]$, $|\overline{B} \cdot I| \leqq 1/2$; hence $|\overline{B} \cdot I|^{-1/2} \geqq 2^{1/2}$; hence $4 \leqq 2^{3/2}|\overline{B} \cdot I|^{-1/2}$. Combining this with the above results, we have

$$1/\rho(I) < 4 + 2^{3/2}|\overline{B} \cdot I|^{-1/2} \leqq 2^{5/2}|\overline{B} \cdot I|^{-1/2},$$

whence

$$\rho(I) > |\overline{B} \cdot I|^{1/2}/2^{5/2}.$$

2. An approximately continuous function whose integral is non-differentiable. For each $t \in B$ we define the function $f_t(x)$ for $x \in [0, 1]$ as follows

$$f_t(x) = \begin{cases} 0 & \text{for } x \leqq t \text{ or } x \in B, \\ |\overline{B} \cdot [t, x]|^{-3/4} & \text{for } x > t \text{ and } x \in \overline{B}. \end{cases}$$

2.1. THEOREM. *For each* $t \in B$, $f_t(x)$ *is an integrable function of* x, *and for* $x_2 \geqq x_1 \geqq t$,

$$\int_{x_1}^{x_2} f_t(x)dx = 4(|\overline{B} \cdot [t, x_2]|^{1/4} - |\overline{B} \cdot [t, x_1]|^{1/4}).$$

Let $z = |\overline{B} \cdot [t, x]|$. Since the intervals of $\overline{B}$ are dense in $[0, 1]$, this defines z as a strictly monotone function of x; hence x is a single-valued function of z, and we may write $f_t[x(z)]$. Now for $x > t$ and $x \in \overline{B}$, $dx = dz$; thus the function $z(x)$ is measure preserving over $\overline{B}$

$\cdot [t, x]$ and hence maps $B \cdot [t, x]$ into a set of measure zero. Therefore, for almost all z, $f_t[x(z)] = z^{-3/4}$; and $dx = dz$ except where $f_t(x) = 0$; so

$$\int_{x_1}^{x_2} f_t(x)dx = \int_{z_1}^{z_2} z^{-3/4} dz = 4(z_2^{1/4} - z_1^{1/4}).$$

2.2. THEOREM. *For $t \in B$, the function*

$$F_t(x) = \int_t^x f_t(u)du$$

is not differentiable with respect to x at $x = t$.

Again letting $z = |\overline{B} \cdot [t, x]|$, and using Theorem 2.1 and Lemma 1.6, we have

$$\frac{F_t(x)}{x - t} = \frac{4z^{1/4}}{x - t} = 4z^{-3/4}\left(\frac{z}{x - t}\right) = 4z^{-3/4}\rho([t, x])$$
$$> 4z^{-3/4}(z^{1/2}/2^{5/2}) = (4z)^{-1/4}.$$

Thus

$$\limsup_{x \to t} \frac{F_t(x)}{x - t} \geq \lim_{z \to 0} (4z)^{-1/4} = \infty.$$

In the next section we shall make further use of the functions $f_t(x)$ and their properties as shown in Theorems 2.1 and 2.2. We might note here, however, that $f_t(x)$ is approximately continuous at t provided $\overline{B}$ has metric density zero at t. This is true for almost all t in B; hence for such t, $f_t(x)$ furnishes a specific example of an approximately continuous function whose integral is not differentiable.

3. A Pettis integral in the space C which is not almost everywhere weakly differentiable. We shall here define a function $\phi(x)$ from $[0, 1]$ to the space C. Our notation will be as follows: For each $x \in [0, 1]$, $\phi(x)$ stands for a continuous function on $[0, 1]$; we denote this continuous function by $\phi_x(t)$. We shall define the functions $\phi_x(t)$ by defining a function $\phi(x, t)$ over the unit square and setting $\phi_x(t) = \phi(x, t)$. We first define $\phi(x, t)$ over a portion of the unit square as follows:

$$\phi(x, t) = \begin{cases} 0 & \text{for } x \in B, \\ f_t(x) & \text{for } t \in B. \end{cases}$$

Since $f_t(x) = 0$ for $x \in B$, these statements are consistent.

3.1. LEMMA. *For a fixed x, $\phi(x, t)$ is continuous in t over B.*

This statement follows immediately from the fact that if one end point of I is fixed, $|\overline{B}\cdot I|^{-3/4}$ is a continuous function of the other end point over any set such that $|I|$ is bounded away from zero. This latter restriction causes no difficulties here. If $x\in B$, $\phi(x, t)=0$; if $x\in\overline{B}$, dist $(x, B)>0$.

We now continue the definition of $\phi(x, t)$. For each $x\in\overline{B}$, let $\phi(x, t)$ be continued linearly over each interval of the set $t\in\overline{B}$. This completes the definition of $\phi(x, t)$ over the entire unit square, and it is clear that for each x, $\phi(x, t)$ is continuous in t over $[0, 1]$.

3.2. THEOREM. *$\phi(x)$ is integrable in the sense of Pettis. For each measurable set $E\subset[0, 1]$, its integral over E is the function*

$$\Phi_E(t) = \int_E \phi(u, t)du.$$

We show this by considering the functions $\phi^{(n)}(x)$ whose values are the continuous functions $\phi_x^{(n)}(t)=\phi^{(n)}(x, t)$ where

$$\phi^{(n)}(x, t) = \begin{cases} \phi(x, t) & \text{for } \phi(x, t) \leqq n, \\ n & \text{for } \phi(x, t) > n. \end{cases}$$

It is easily seen that for each $t\in[0, 1]$, each $\phi^{(n)}(x, t)$ is bounded and continuous in x over $\overline{B}$. Thus[3] each $\phi^{(n)}(x)$ is weakly continuous over $\overline{B}$. Since $\phi^{(n)}(x)=\theta$ for $x\in B$, it is clear that each $\phi^{(n)}(x)$ is weakly measurable. Since C is a separable space, it follows[4] that each $\phi^{(n)}(x)$ is measurable. Now each $\phi^{(n)}(x)$ is bounded, hence Bochner integrable, hence Pettis integrable, therefore integrable with respect to each of the linear functionals $\gamma_t[\phi^{(n)}(x)]=\phi_x^{(n)}(t)$; thus

$$\Phi^{(n)}(E) = \int_E \phi^{(n)}(u)du$$

is the continuous[5] function

$$\Phi_E^{(n)}(t) = \int_E \phi^{(n)}(u, t)du.$$

Clearly for each x and each t,

$$\lim_{n\to\infty} \phi_x^{(n)}(t) = \phi_x(t);$$

[3] See [1, p. 224, Theorem 8].

[4] See [3, Theorem 1.1].

[5] An independent proof of continuity of $\Phi_E^{(n)}(t)$ is unnecessary. Since $\phi^{(n)}(x)$ is Pettis integrable, it is integrable to an element of C; and the set $\{\gamma_t\}$ of linear functionals defines this element uniquely.

furthermore this approximation is monotone in n. Thus $\|\phi^{(n)}(x)-\phi(x)\|$ is bounded in n for each x; hence[6] $\phi^{(n)}(x)\to\phi(x)$ weakly for each x. It now follows[7] that $\phi(x)$ is Pettis integrable provided the sequence $\{\Phi^{(n)}(E)\}$ converges with respect to the norm in C; that is, provided $\{\Phi_E^{(n)}(t)\}$ converges uniformly in t. We shall complete the proof of Theorem 3.2 by showing that for each measurable $E\subset[0, 1]$

$$\Phi_E(t) = \int_E \phi(u, t)du$$

exists for each t and that this function is the uniform limit of the sequence $\{\Phi_E^{(n)}(t)\}$.

To show that $\Phi_E(t)$ exists is trivial. For $t\in B$, this follows from Theorem 2.1. For each x, $\phi(x, t)$ is extended linearly over each interval of $t\in\overline{B}$; hence for $t\in\overline{B}$, $\phi(x, t)\leqq\phi(x, t_1)+\phi(x, t_2)$ where t_1 and t_2 are each in B. This completes the proof of integrability.

Now with each $t\in[0, 1]$ we associate two numbers t_1 and t_2 as follows: t_1 is the greatest number such that $t_1\in B$ and $t_1\leqq t$; t_2 is the smallest number such that $t_2\in B$ and $t_2\geqq t$. Geometrically this means that if $t\in B$, $t_1=t=t_2$, while if $t\in\overline{B}$, t_1 and t_2 are the left and right points respectively of the interval of $\overline{B}$ in which t is located.

Now for $t_1\leqq x<t_2$, $\phi(x, t)\leqq\phi(x, t_1)$ while for $x\geqq t_2$, $\phi(x, t)\leqq\phi(x, t_2)$. Thus for any given $t\in[0, 1]$, it is possible to have $\phi(x, t)>n$ only for those values of x for which either

$$t_1 \leqq x < t_2 \text{ and } |\overline{B}\cdot[t_1, x]| < n^{-4/3}$$

or

$$x \geqq t_2 \text{ and } |\overline{B}\cdot[t_2, x]| < n^{-4/3}.$$

Outside these two intervals $\phi(x, t)-\phi^{(n)}(x, t)=0$; hence if we denote these intervals by I_1 and I_2, we have

$$\begin{aligned}\int_E [\phi(x, t) - \phi^{(n)}(x, t)]dx &\leqq \int_{I_1} \phi(x, t)dx + \int_{I_2} \phi(x, t)dx \\ &\leqq \int_{I_1} \phi(x, t_1)dx + \int_{I_2} \phi(x, t_2)dx \\ &< 2\int_0^{n^{-4/3}} z^{-3/4}dz = 8n^{-1/3}.\end{aligned}$$

[6] See [1, p. 224, Theorem 8].

[7] See [3, Theorem 4.1].

Thus, clearly, $\lim_{n\to\infty}\Phi_{E^{(n)}}(t)=\Phi_E(t)$ uniformly in t. This completes the proof of Theorem 3.2.

3.3. THEOREM. *If $x_0\in B$, $\Phi(x)=\int_0^x\phi(u)du$ is not weakly differentiable at x_0.*

By Theorem 2.2, it fails to be differentiable at x_0 with respect to the linear functional $\gamma_{x_0}[\Phi(x)]=\Phi_x(x_0)$.

4. **Extension to other spaces of continuous functions.** The function $\phi(x)$ of §3 may be used as the basis for the construction of a large set of examples as follows:

4.1. THEOREM. *If Ω is a compact metric space containing non-denumerably many points and if $C(\Omega)$ is the Banach space of all continuous functionals on Ω, then there is a function $\psi(x)$ from the unit interval to $C(\Omega)$ such that $\psi(x)$ is Pettis integrable but $\Psi(E)=\int_E\psi(x)dx$ fails to be weakly differentiable on a set of positive measure.*

Since Ω is non-denumerable, it contains a perfect set. This perfect set is a complete metric space which is dense in itself and hence contains a homeomorph Π of the Cantor set B.[8]

Let $B=h(\Pi)$ be the homeomorphic mapping of Π into B. Then $h(\omega)$ is a continuous function defined over Π, assuming values between 0 and 1, and assuming for some $\omega\in\Pi$ each value in the set B. Let $H(\omega)$ be a continuous extension[9] of $h(\omega)$ over the whole of Ω with $0\leqq H(\omega)\leqq 1$.

Now for each $t\in[0, 1]$ we define

$$K(t) = \underset{\omega}{\mathcal{E}}\{H(\omega) = t\}.$$

It should be noted that although for some t, $K(t)$ may be vacuous, for each $t\in B$, $K(t)$ contains at least one point.

Referring back to the functions $\phi_x(t)$ of §3, we now define

$$\psi_x(\omega) = \phi_x(t) \quad \text{for} \quad \omega \in K(t).$$

It follows from the continuity in t of each function $\phi_x(t)$ and from the continuity of $H(\omega)$ that for each $x\in[0, 1]$, $\psi_x(\omega)$ is continuous over Ω. For each $x\in[0, 1]$ we now let $\psi(x)$ be the element $\psi_x(\omega)$ of $C(\Omega)$.

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[8] See [**2**, p. 228]. The author is indebted to the referee for the suggestion that non-denumerability of Ω is sufficient to insure the existence of Π.

[9] See [**2**, p. 211]. In connection with our remark in the introduction that we have a specific construction applicable to the more general spaces, it should be noted that this extension theorem is not merely an existence proof. A definite formula for the extension is given.

That $\psi(x)$ has the required properties may be seen as follows: To show integrability, we note that for each $\omega \in \Omega$, $\psi_x(\omega)$ is identical (as a function of x) with $\phi_x(t)$ for some $t \in [0, 1]$. Then noting that Banach's criterion for weak convergence[10] applies to the space $C(\Omega)$, we apply the proof of Theorem 3.2. To show non-differentiability, we note that for each $t \in B$ there is an $\omega \in \Omega$ such that $\psi_x(\omega) = \phi_x(t)$ for all $x \in [0, 1]$. The proof of Theorem 3.3 then applies.

References

1. S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, Warsaw, 1932.
2. C. Kuratowski, *Topologie*, I, Monografje Matematyczne, Warsaw, 1933.
3. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 277–304.
4. R. S. Phillips, *Integration in a convex linear topological space*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 114–145.

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[10] This is used in the proof of Theorem 3.2. See footnotes 3 and 6.