COUNTABLE CONNECTED SPACES

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Introduction. Let $\mathfrak{S}$ be the class of all countable and connected perfectly separable Hausdorff spaces containing more than one point. It is known that an $\mathfrak{S}$-space cannot be regular or compact. Urysohn, using a complicated identification of points, has constructed the first example of an $\mathfrak{S}$-space.[1] Two $\mathfrak{S}$-spaces, X and X^*, more simply constructed and not involving identifications, are presented here. The space X^* is a connected subspace of X and contains a dispersion point; that is, the subspace formed from X^* by removing this one point is totally disconnected.

1. **Sequences.** The null sequence or any finite sequence of positive integers will hereafter be called more briefly a sequence. The null sequence or a sequence having an even number of elements is said to be even and a sequence having an odd number of elements is said to be odd. A sequence will usually be denoted by a lower case Greek letter: an arbitrary sequence by α, β, or γ; an arbitrary even sequence by λ, μ, or ν; the null sequence by o. A positive integer will be denoted by a lower case italic letter (not x, y, or z), which may also serve to represent the sequence consisting of that single integer.

The relation $\alpha \geqq i$ signifies that $a \geqq i$ for every element a of α, and $\alpha < i$ that $a < i$ for every element a of α. The null sequence vacuously satisfies both $o \geqq i$ and $o < i$.

The sequence formed by adjoining β to the end of α is denoted by $\alpha\beta$.

Definition. *The relation $\beta \supset_i \alpha$ is to mean that a sequence α' exists such that $\beta = \alpha\alpha'$ and $\alpha' \geqq i$.*

Some immediate consequences of the preceding definitions are:

1.1. $\alpha \supset_i \alpha$.

1.2. *If $\beta \supset_j \alpha$ and $j \geqq i$, then $\beta \supset_i \alpha$.*

1.3. *If $\gamma \supset_i \beta$ and $\beta \supset_i \alpha$, then $\gamma \supset_i \alpha$.*

1.4. *If $\gamma \supset_a \alpha$ and $\gamma \supset_b \beta$, then $\beta \supset_a \alpha$ or $\alpha \supset_b \beta$.*

Proof. Let $\gamma \supset_a \alpha$ and $\gamma \supset_b \beta$; then sequences α', β' exist such that

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[1] Paul Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. vol. 94 (1925) pp. 262–295; see pp. 274–283 for the example.

$\gamma=\alpha\alpha'$, $\alpha'\geqq a$, and $\gamma=\beta\beta'$, $\beta'\geqq b$. Since $\alpha\alpha'=\beta\beta'$, there exists a sequence α'' such that $\beta=\alpha\alpha''$ or a sequence β'' such that $\alpha=\beta\beta''$. If $\beta=\alpha\alpha''$, then $\alpha\alpha'=\beta\beta'=\alpha\alpha''\beta'$; hence $\alpha'=\alpha''\beta'$. But $\alpha'\geqq a$, so $\alpha''\geqq a$ and consequently $\beta\supset_a\alpha$. Similarly, if $\alpha=\beta\beta''$, then $\alpha\supset_b\beta$.

2. **Points in** X. The space X shall consist of two disjoint subsets: Y, the set of all *even* sequences; and Z, the set of all ordered pairs $\{k, (\mu, \nu)\}$ composed of a positive integer k and a set $(\mu, \nu)=(\nu, \mu)$ of *even* sequences μ and ν. Hereafter a point μ in Y will be denoted by $y(\mu)$ and a point $\{k, (\mu, \nu)\}$ in Z by $z_k(\mu, \nu)$. Evidently X is countable.

The neighborhoods in X will be formed from certain subsets $Y_i(\alpha)$ of Y, defined for every positive integer i and every sequence α.

DEFINITION. *$Y_i(\alpha)$ is the set of all points $y(\mu)$ such that $\mu\supset_i\alpha$.*

Some properties of these sets are:

2.1. $y(\mu)\in Y_i(\mu)$.

2.2. *If $j\geqq i$, then $Y_j(\alpha)\subset Y_i(\alpha)$.*

2.3. *If $y(\mu)\in Y_i(\alpha)$, then $Y_i(\mu)\subset Y_i(\alpha)$.*

2.4. *$Y_a(\alpha)Y_b(\beta)\neq 0$ is equivalent to: $\beta\supset_a\alpha$ or $\alpha\supset_b\beta$.*

PROOF. If the set $Y_a(\alpha)Y_b(\beta)$ contains a point $y(\mu)$, then $\mu\supset_a\alpha$ and $\mu\supset_b\beta$; therefore, $\beta\supset_a\alpha$ or $\alpha\supset_b\beta$.

Now, if $\beta\supset_a\alpha$, define $m=\max(a, b)$, $\nu=\beta$ if β is even, $\nu=\beta m$ if β is odd. Thus ν is even and $\nu\supset_b\beta$ so $y(\nu)\in Y_b(\beta)$. Moreover $\nu\supset_a\beta\supset_a\alpha$, hence $\nu\supset_a\alpha$ so $y(\nu)\in Y_a(\alpha)$. Therefore $Y_a(\alpha)Y_b(\beta)\neq 0$. Similarly $Y_a(\alpha)Y_b(\beta)\neq 0$ if $\alpha\supset_b\beta$.

COROLLARY. *If $\alpha\neq\beta$ and $\alpha\beta<i$, then $Y_i(\alpha)Y_i(\beta)=0$.*

To every point $z=z_k(\mu, \nu)$ a unique positive integer $q(z)=q_k(\mu, \nu)$ is assigned as follows. The set of all sets (μ, ν) of even sequences μ and ν, being countable and infinite, can be mapped onto the set of positive integral primes by some 1-1 mapping $p(\mu, \nu)$. Define

$$q_k(\mu, \nu) = [p(\mu, \nu)]^k$$

According to the unique factorization theorem of arithmetic, q is a 1-1 mapping of the point set Z onto a subset of the positive integers. Moreover, since the infinite sequence of positive integers $q_k(\mu, \nu)$ for $k=1, 2, \cdots$ is strictly increasing, $q_k(\mu, \nu)\to\infty$ as $k\to\infty$.

3. **Neighborhoods in X.** For every point x in X and every positive integer i, a neighborhood $V_i x$ of x is now defined.

DEFINITION.

$$V_i y(\mu) = Y_i(\mu);$$

$$V_i z_k(\mu, \nu) = z_k(\mu, \nu) + Y_i(\mu q) + Y_i(\nu q), \qquad q = q_k(\mu, \nu).$$

Under this definition of neighborhood X forms a Hausdorff topological space; that is, X satisfies the following neighborhood axioms.

Axiom 1. To every point x in X there corresponds at least one neighborhood of x; every neighborhood of x contains x by 2.1 or by definition.

Axiom 2. If $V_i x$ and $V_j x$ are two neighborhoods of x, a neighborhood $V_m x$ of x exists such that $V_m x \subset V_i x V_j x$. Indeed, if $m = \max(i, j)$, then $V_m x = V_i x V_j x$ by 2.2.

Axiom 3. If $V_i x$ contains a point $y(\mu)$, there exists a neighborhood of $y(\mu)$ contained in $V_i x$. By 2.3 such a neighborhood is $V_i y(\mu)$.

Axiom 4H. Every two distinct points x, x' in X are *Hausdorff- or H-separable*; that is, there exist neighborhoods $V_i x$ of x and $V_i x'$ of x' such that $V_i x V_i x' = 0$. The intersection $V_i x V_i x'$ can be reduced to the sum of at most four intersections, each of the form $Y_i(\alpha) Y_i(\alpha')$. If α, α' are both even, then $\alpha \neq \alpha'$ since $x \neq x'$. And also $\alpha \neq \alpha'$, if α, α' are both odd; for then even sequences μ, μ' and positive integers q, q' exist such that $\alpha = \mu q$, $\alpha' = \mu' q'$, and, since $x \neq x'$, $q \neq q'$. Thus, according to the corollary of 2.4, $Y_i(\alpha) Y_i(\alpha') = 0$ when i is chosen so that $\alpha\alpha' < i$. An integer i then exists for which $V_i x V_i x' = 0$.

Thus X is a nondegenerate countable Hausdorff space. Evidently X is also perfectly separable.

4. **Connectedness of X.** Two distinct points x, x' in a space E are said to be *$\overline{H}$-separable* provided neighborhoods V of x and V' of x' exist such that $\overline{V}\overline{V}' = 0$; otherwise, the points x, x' are said to be *$\overline{H}$-inseparable.* A single point is also said to be $\overline{H}$-inseparable if it is $\overline{H}$-inseparable with every other point in E.

A space E containing an $\overline{H}$-inseparable point x is connected; for otherwise E could be covered by two non-null disjoint isolated (open and closed) sets V, V', one of which contains x; but this would imply the contradiction

$$0 = VV' = \overline{V}\overline{V}' \neq 0.$$

Moreover, if E is a Hausdorff space, then no point of E satisfies the regularity axiom, or, more briefly, is regular. For let x' be any point in E distinct from x. Since x, x' are H-separable in E, there exist disjoint neighborhoods V of x and V' of x'; consequently

$$V\overline{V}' = 0 = \overline{V}V'.$$

If x were a regular point of E, then a neighborhood U of x would exist such that $V \supset \overline{U}$, so

$$0 = V\overline{V}' \supset \overline{U}\overline{V}' \neq 0.$$

Similarly, if x' were a regular point of E, then a neighborhood U' of x' would exist such that $V' \supset \overline{U}'$, so

$$0 = \overline{V}V' \supset \overline{V}\overline{U}' \neq 0.$$

By considering the sets $Y_i(\alpha)$ every point in the space X is now shown to be $\overline{H}$-inseparable. Hence X is connected and no point of X is regular.

Definition. *$Z_i(\alpha)$ is the set of all points $z_k(\mu, \nu)$ such that $\mu q \supset_i \alpha$ or $\nu q \supset_i \alpha$, $q = q_k(\mu, \nu)$.*

4.1. $\overline{Y}_i(\alpha) = Y_i(\alpha) + Z_i(\alpha)$.

Proof. The following equivalent statements show that $Y\overline{Y}_i(\alpha) = Y_i(\alpha)$:

$y(\mu) \in \overline{Y}_i(\alpha)$.
For all j: $V_j y(\mu) Y_i(\alpha) \neq 0$.
For all j: $Y_j(\mu) Y_i(\alpha) \neq 0$.
For all j: $\alpha \supset_j \mu$ or $\mu \supset_i \alpha$.
$\mu \supset_i \alpha$.
$y(\mu) \in Y_i(\alpha)$.

The following equivalent statements show that $Z\overline{Y}_i(\alpha) = Z_i(\alpha)$, where $q_k(\mu, \nu)$ has been abbreviated to q:

$z_k(\mu, \nu) \in \overline{Y}_i(\alpha)$.
For all j: $V_j z_k(\mu, \nu) Y_i(\alpha) \neq 0$.
For all j: $[Y_j(\mu q) + Y_j(\nu q)]\ Y_i(\alpha) \neq 0$.
For all j: $Y_i(\mu q) Y_i(\alpha) \neq 0$ or $Y_i(\nu q) Y_i(\alpha) \neq 0$.
For all j: $\alpha \supset_j \mu q$ or $\mu q \supset_i \alpha$ or $\alpha \supset_j \nu q$ or $\nu q \supset_i \alpha$.
$\mu q \supset_i \alpha$ or $\nu q \supset_i \alpha$.
$z_k(\mu, \nu) \in Z_i(\alpha)$.

4.2. *$Z_a(\alpha) Z_b(\beta) \neq 0$; hence every two distinct points in X are $\overline{H}$-inseparable.*

Proof. Evidently there exist even sequences μ, ν such that $\mu \supset_a \alpha$ and $\nu \supset_b \beta$. And since $q_k(\mu, \nu) \to \infty$ as $k \to \infty$ a positive integer k exists for which

$$q = q_k(\mu, \nu) \geqq \max (a, b).$$

Therefore $\mu q \supset_a \alpha$ and $\nu q \supset_b \beta$; so $z_k(\mu, \nu) \in Z_a(\alpha) Z_b(\beta)$.

Thus X is an $\mathfrak{S}$-space whose every point is $\overline{H}$-inseparable.

5. **The space** X^*. Let X^* be the relative subspace of X formed by removing from X all points $z_k(\mu, \nu)$ except those of the form $z_k(\mu, o)$, $\mu \neq o$. Notice that every X^*-neighborhood of a point in X^* is also an X-neighborhood of that point. The argument of 4.2 shows that the set $Z_a(\alpha) Z_b(o)$ contains a point of X^*. The point $y(o)$ is then an $\overline{H}$-inseparable point of X^*. Thus X^*, being a nondegenerate connected subspace of an $\mathfrak{S}$-space, is also an $\mathfrak{S}$-space.

6. **The space** X^{**}. Let X^{**} be the relative subspace of X^* formed by removing from X^* the single point $y(o)$. This point is a dispersion point of X^*; for the following recursive construction of isolated subsets in the space X^{**} shows that X^{**} is totally disconnected.

DEFINITION. *For every non-null even sequence* λ *and every positive integer* i *such that* $\lambda < i$ *let*

$$X_i(\lambda) = \sum_{n=1}^{\infty} [Y_i^n(\lambda) + Z_i^n(\lambda)],$$

the sets $Y_i^n(\lambda)$ *and* $Z_i^n(\lambda)$ *being recursively defined as follows:*

$Y_i^n(\lambda)$ *is the set of all points* $y(\mu)$ *such that* $\mu \supset_i \alpha^n$, *where* $\alpha^n = \lambda$ *if* $n = 1$, *and* $\alpha^n = q(z)$ *for some* $z \in Z_i^{n-1}(\lambda)$ *if* $n > 1$;

$Z_i^n(\lambda)$ *is the set of all points* $z = z_k(\mu, o)$ *such that* $y(\mu) \in Y_i^n(\lambda)$ *and* $q(z) \geqq i$.

6.1. $V_i x \subset X_i(\lambda)$ *for all* $x \in X_i(\lambda)$; *hence* $X_i(\lambda)$ *is open in* X^{**}.

PROOF. Let $y(\mu) \in Y_i^n(\lambda)$; then $\mu \supset_i \alpha^n$. If $y(\nu) \in V_i y(\mu)$, then $\nu \supset_i \mu \supset_i \alpha^n$, so $y(\nu) \in Y_i^n(\lambda)$.

Let $z = z_k(\mu, o) \in Z_i^n(\lambda)$; then $\mu \supset_i \alpha^n$ and $q(z) \geqq i$. If $y(\nu) \in V_i z$, then $\nu \supset_i \mu q(z)$ or $\nu \supset_i q(z)$. Now $\nu \supset_i \mu q(z)$, $q(z) \geqq i$, implies that $\nu \supset_i \mu \supset_i \alpha^n$ and hence that $y(\nu) \in Y_i^n(\lambda)$. And $\nu \supset_i q(z)$, $z \in Z_i^n(\lambda)$, implies that $y(\nu) \in Y_i^{n+1}(\lambda)$.

6.2. $V_i x X_i(\lambda) = 0$ *for all* $x \notin X_i(\lambda)$; *hence* $X_i(\lambda)$ *is closed in* X^{**}.

PROOF. Let $y(\mu) \notin X_i(\lambda)$. Suppose the set $V_i y(\mu) X_i(\lambda)$ contains a point $y(\nu)$; then $\nu \supset_i \mu$ and $\nu \supset_i \alpha^n$. Therefore $\mu \supset_i \alpha^n$ or $\alpha^n \supset_i \mu$. Now $\alpha^n \not\supset_i \mu$, since $\alpha^1 = \lambda < i$ and since α^n is a single integer if $n > 1$. Hence $\mu \supset_i \alpha^n$, so $y(\mu) \in Y_i^n(\lambda)$—a contradiction.

Let $z = z_k(\mu, o) \notin X_i(\lambda)$. Suppose the set $V_i z X_i(\lambda)$ contains a point

$y(\nu)$; then $\nu \supset_i \mu q(z)$ or $\nu \supset_i q(z)$, and $\nu \supset_i \alpha^n$. Therefore

$$\mathfrak{A}^n: \alpha^n \supset_i \mu q(z) \text{ or } \mu q(z) \supset_i \alpha^n \text{ or } \alpha^n \supset_i q(z) \text{ or } q(z) \supset_i \alpha^n.$$

Now $\lambda \neq o$, $\lambda < i$, and $\lambda = \alpha^1$; $\mathfrak{A}^1$ then reduces to $\mu q(z) \supset_i \lambda$; so $\mu \supset_i \lambda$, $q(z) \geqq i$, and consequently $z \in Z_i^1(\lambda)$—a contradiction. If $n > 1$, then $\alpha^n = q(z') \geqq i$ for some $z' \in Z_i^{n-1}(\lambda)$, so $\mathfrak{A}^n$ reduces to: $\mu q(z) \supset_i q(z')$, $\mu \neq o$; or $q(z) = q(z')$. Now $\mu q(z) \supset_i q(z')$, $\mu \neq o$, implies that $\mu \supset_i q(z')$, $q(z) \geqq i$, and hence that $z \in Z_i^n(\lambda)$—a contradiction. And $q(z) = q(z')$ implies that $z = z' \in Z_i^{n-1}(\lambda)$—also a contradiction.

The sets $X_i(\lambda)$ are then isolated subsets of X^{**} for $\lambda \neq o$, $\lambda < i$. Notice that

$$\begin{aligned} x &= y(\lambda) \in X_i(\lambda), \\ x &= y(\lambda') \notin X_i(\lambda) && \text{if } \lambda' \neq \lambda \text{ and } \lambda' < i, \\ x &= z_k(\lambda, o) \in X_i(\lambda) && \text{if } q(x) \geqq i, \\ x' &\notin X_i(\lambda) && \text{if } x' \in Z \text{ and } q(x') < i. \end{aligned}$$

Now there exists for any two distinct points x, x' in X^{**} an isolated set $X_i(\lambda)$ containing x but not x': if $x = y(\lambda)$, $x' = y(\lambda')$, choose i so that $\lambda\lambda' < i$; if $x = y(\lambda)$ and $x' \in Z$, choose i so that $\lambda q(x') < i$; and if $x = z_k(\lambda, o)$ and $x' \in Z$, choose $i = q(x)$, then $q(x') < i$ and $\lambda < i$, since it may be assumed that $q(x') < q(x)$ and since the mapping p can be selected so that $\mu\nu < p(\mu, \nu)$.

Thus the space X^{**} is totally disconnected. In particular, every two distinct points in X^{**} are $\overline{H}$-separable; hence $y(o)$ is the only $\overline{H}$-inseparable point of X^*.

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