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THE UNIVERSITY OF TEXAS

ON ISOMETRIES OF SQUARE SETS

PAUL J. KELLY

1. Introduction. It is not fully known under what conditions the isometry of two square, metric sets, say E^2 and F^2, implies the isometry of E and F. Using the notion of order two self-isometries, this paper gives conditions sufficient to imply E isometric to F when E^2 and F^2 are finite and are metrized under any one of a fairly extensive class of functions. The basic ideas are first applied to non-square sets to yield a more general theorem which is then applied to the inverse square problem.

2. Definitions. A set is called metric if to every pair of its elements, a and b, there corresponds a real, non-negative number, which is independent of the order of a and b, zero if and only if a equals b, and which satisfies the triangle law.

Two metric sets are isometric (written "$\equiv$") if there is a one-to-one transformation of one set on the other in which the metric number associated with any pair is the same as that associated with the transformed pair.

A non-identity mapping of a set on itself, which is an isometry, and which leaves each element of the set invariant or else interchanges it with another, is called a self-isometry of order two. Any subset on which the self-isometry is the identity is said to be left pointwise invariant.

THEOREM 1. *Assume $A \equiv B$ under a mapping T, where A and B are finite metric sets. Let A and B have self-isometries of order two under mappings R and S respectively and let A_1 and B_1 denote respectively the maximum subsets left pointwise invariant. If A_1 has no self-isometry of order two, and has at least as many elements as B_1, then $A_1 \equiv B_1$ and there*

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exists a composition of R, S, T and T^{-1} which maps A isometrically on B and carries A_1 into B_1.

PROOF. Starting with the set A_1 the following sequence of sets outlined is obtained by transforming A_1 by T, the set obtained by S, this set by T^{-1}, and this set by R, and so on repeating cyclically the transformations T, S, T^{-1}, R.

	Column 1		Column 2	
	A_1	$\longrightarrow$	$T(A_1)$	(1, a)
(2, a)	$T^{-1}ST(A_1)$	$\longleftarrow$	$ST(A_1)$	(1, b)
(2, b)	$RT^{-1}ST(A_1)$	$\longrightarrow$	$TRT^{-1}ST(A_1)$	(3, a)
		$\longleftarrow$	$STRT^{-1}ST(A_1)$	(3, b)
(2n, a)				
(2n, b)		$\longrightarrow$		(2n + 1, a)
				(2n + 1, b)

The notation at the side is such that set (n, x), $x=a$ or b, is in B if n is odd and in A if n is even. From the construction and the nature of R and S, the following relations are easily verified: $R(2n, a)=(2n, b)$, $R(2n, b)=(2n, a)$, $S(2n+1, a)=(2n+1, b)$, $S(2n+1, b)=(2n+1, a)$, $T(2n, b)=(2n+1, a)$, $T^{-1}(2n+1, b)=(2n+2, a)$.

(1) Assume no set in column 2 is the set B_1.

(2) Since all sets in both columns are isometric to A_1, isometry being transitive, and since A_1 has as many elements as B_1, (1) implies that no set in column 2 is a subset of B_1.

(3) For any n, $x=a$ or b, $S(2n+1, x)\neq(2n+1, x)$. Since S is the identity mapping only on B_1 and since, from (1) and (2), $(2n+1, x)$ is not B_1 or a subset of it, $S(2n+1, x)=(2n+1, x)$ would mean that $(2n+1, x)$ had a self-isometry of order two. This, together with $A_1\equiv(2n+1, x)$, would imply A_1 had a self-isometry of order two, contradicting the given conditions.

(4) For any n, no two sets of column 1 up to and including $(2n, a)$ are identical. The proof is by induction.

(4.1) Statement (4) holds for $n=1$, since $A_1=(2, a)$ would give $T(A_1)=T(2, a)$ or $(1, a)=(1, b)$, contradicting (3).

(4.2) Assume (4) holds for $n=k$.

(4.3) Since R is the identity only on A_1 and since $(2k, a)$ is not a subset of A_1, being isometric to it, and is not equal to A_1, from (4.2), then $R(2k, a)=(2k, a)$ would imply that $(2k, a)$ had a self-isometry of order two, and hence that A_1 did also. Therefore $R(2k, a) \neq (2k, a)$, that is $(2k, b) \neq (2k, a)$. This, in turn, implies $(2k, b) \neq A_1$.

(4.4) For $i<k$, $x=a$ or b, $(2k, b) \neq (2i, x)$. From $(2k, b) = (2i, x)$ would follow $R(2k, b) = R(2i, x)$, that is $(2k, a) = R(2i, x)$, which for $i<k$ would contradict (4.2).

(4.5) From (4.2), (4.3), and (4.4) no two sets of column 1 up to and including $(2k, b)$ are identical. This, with the one-to-oneness of T, implies that no two sets of column 2 up to and including $(2k+1, a)$ are identical.

(4.6) From (3), $(2k+1, b) \neq (2k+1, a)$.

(4.7) For $i<k$, $x=a$ or b, $(2k+1, b) \neq (2i+1, x)$. For, from $(2k+1, b) = (2i+1, x)$ would follow $S(2k+1, b) = S(2i+1, x)$, that is $(2k+1, a) = S(2i+1, x)$, which for $i<k$ would contradict (4.5).

(4.8) From (4.6) and (4.7) no two sets of column 2 up to and including $(2k+1, b)$ are identical. This, with the one-to-oneness of T^{-1}, implies that no two sets of column 1 up to and including $(2(k+1), a)$ are identical, and completes the induction establishing (4).

(5) Since (4) implies the existence of an unlimited number of distinct subsets in the finite set A, it is clearly a contradiction reached through assuming (1). Therefore (1) is false and B_1 must occur in column 2 and be isometric to A_1. The remainder of the theorem follows from the fact that the sequence of sets can be started with A rather than A_1.

If A and B are the same set and T is replaced by the identity, Theorem 1 reduces to the following result:

THEOREM 2. *Let A be a finite metric set and let A_1 and B_1 be the maximum subsets left pointwise invariant under two distinct self-isometries, R and S, of order two. If A_1 has no self-isometry of order two and has at least as many elements as B_1, then $A_1 \equiv B_1$ and there is a composition of R and S which maps A isometrically on itself and carries A_1 into B_1.*

3. **Definitions concerning square sets.** Let E be a finite metric set with elements $x_1, x_2, \cdots, x_n$ and metric ρ_E. By E^2 is meant the set of couples obtained from the cartesian product of E with itself.

In E^2 the subset of couples (x_i, x_i), $i=1, 2, \cdots, n$, is called the diagonal set.

The reflection mapping, R, of E^2 on itself is defined by $R(x_i, x_j) = (x_j, x_i)$.

If a metric ρ_{E^2} is defined on the elements of E^2 it is called a metric of class α if, in addition to making E^2 a metric set, it has the following properties:

(1) For any two points of E^2, $P_1:(x_i, x_j)$, $P_2:(x_k, x_l)$, $\rho_{E^2}(P_1, P_2) = f(X_1, X_2)$ where $X_1 = \rho_E(x_i, x_k)$, $X_2 = \rho_E(x_j, x_l)$.

(2) $f(X_1, X_2) = f(X_2, X_1)$.

(3) There exists a constant M associated with f, such that whenever $X_1 = X_2$, then $f(X_1, X_2) = MX_1$.

THEOREM 3. *Let E and F be finite metric sets, and let E^2 and F^2 be metrized under the same class α metric. If either the diagonal set of E^2 or that of F^2 has no self-isometry of order two, then $E^2 \equiv F^2$ implies $E \equiv F$.*

PROOF. Let R and S denote respectively the reflection mappings of E^2 and F^2 on themselves. From the definition of reflection and from property 2 of a class α metric, the mappings R and S establish self-isometries of order two in which the diagonal sets alone are left pointwise invariant. The two diagonal sets also have the same number of elements because $E^2 \equiv F^2$. From Theorem 1, then, with E^2 and F^2 playing the roles of A and B, and with the diagonal sets as A_1 and B_1, it follows that the diagonal set of E^2 is isometric to that of F^2. This isometry together with property 3 of a class α metric implies $E \equiv F$.

UNIVERSITY OF WISCONSIN