QUADRATIC FORMS

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This note concerns itself with the concept of the signature of a real quadratic form and with Sylvester's classical theorem of "inertia." The standard treatment is elementary enough and involves merely simple algebraic algorithms. The usual extensions have been in the direction of replacing the real number field by other fields. Roughly speaking, the viewpoint has been algebraic. The viewpoint developed below is different. It leads to an interesting identification of the signature with a certain topological invariant. This suggests that the corresponding invariant for other forms may be taken as a natural generalization of the signature. Moreover, although the signature and rank are sufficient to characterize quadratic forms, it would seem that for more general forms the topological aspects provide a basis for defining other numerical invariants.

Write

$$Q = \sum_{0}^{k} x_i^2 - \sum_{k+1}^{n+1} x_j^2, \qquad k \geqq 0.$$

Since we may equally well use $-Q$, there is no restriction in assuming $k \leqq n-k$. If $k \geqq 0$, the equation $Q=0$ defines a hypersurface in an $(n+1)$-dimensional real projective space, with coordinates $x_0:x_1: \cdots :x_{n+1}$. Let R_j, $j=0, \cdots, n$, be the mod 2 Betti numbers.

THEOREM. *The signature of Q is $n+2-\sum_0^n R_j$. The rank and the signature are invariant under real (nonsingular) projective transformations.*

It is worth noting the appearance of the sum of the Betti numbers and not the alternating sum. Indeed so far as the writer is aware, this is the first time that $\sum R_j$ has entered in a natural way.

The second assertion of the theorem is, of course, a trivial reflection of the invariance under homeomorphism of the Betti groups and the dimension (n) which is 2 less than the rank. To establish the first part we need the explicit values of the Betti numbers R_j. We can calculate these for $k>0$ by making use of results of Steenrod and Tucker,[1] for instance. An obvious conclusion from their work is that the mod 2 homology groups for $Q=0$ are isomorphic to those of $P^k \times S^{n-k}$ where

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[1] N. E. Steenrod and A. W. Tucker, *Real n quadrics as sphere bundles*, Bull. Amer. Math. Soc. vol. 47 (1941) p. 399.

P^k is a k-dimensional projective space and S^{n-k} is an $(n-k)$-dimensional sphere.

Write ${}_1R_j$ and ${}_2R_j$ for the j-dimensional mod 2 Betti numbers of P^k and S^{n-k} respectively. It is well known that

$$ {}_1R_j = 1, \qquad j = 0, \cdots, k, \tag{1a} $$

$$ {}_2R_i = \delta_{0i} + \delta_{n-k\,i}, \qquad i = 0, \cdots, n-k, \tag{1b} $$

where, of course, δ_{ij} is the Kronecker symbol. The Kunneth relations[2] lead to

$$ \sum_0^n R_j = \sum_0^k {}_1R_j \sum_0^{n-k} {}_2R_j = 2(k+1). \tag{2} $$

For $k=0$ the observation that $Q=0$ is a sphere leads immediately to a result formally consistent with equation 2, namely

$$ \sum_0^n R_j = 2. \tag{2a} $$

The signature of Q is evidently $n-2k$. In view of equations 2 and 2a, this is the value of $n+2-\sum_0^n R_j$. The theorem is therefore established.

The case of definite forms is not compassed by the theorem, for then the configuration $Q=0$ has no meaning in projective geometry. Of course, in this case we need not differentiate between the signature and the rank. Alternatively we may agree to identify the signature of Q with that of $Q+x_{-1}^2-x_{-2}^2$ with transformations restricted to the variables entering Q alone.

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[2] S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, p. 141.