HERMITIAN QUADRATIC FORMS IN A QUASI-FIELD

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1. Introduction. E. Witt[1] proved the following theorem concerning quadratic forms in a fairly general field:

THEOREM 1. *Let* $f_1 = ax_1^2 + \phi_1(x_2, \cdots, x_n)$ *and* $f_2 = ax_1^2 + \phi_2(x_2, \cdots, x_n)$ *be quadratic forms whose coefficients lie in a given field F in which* $2 \neq 0$. *Then the equivalence in F of* f_1 *and* f_2 *implies that of* ϕ_1 *and* ϕ_2.

It is our purpose here to generalize this theorem to any quasi-field (a field, except that multiplication may not be commutative) on which is defined a conjugate operation of period 2 with the usual properties

$$\overline{a+b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b} \cdot \bar{a}.$$

Well known examples are any field with $\bar{a} = a$; the field of complex numbers with the usual complex conjugate; the system of quaternions with real coefficients and the usual conjugate. The analogue in a quasi-field of quadratic form in a field is the hermitian quadratic form

$$f = \bar{x}'Ax = \sum_{i,j=2}^{n} \bar{x}_i a_{ij} x_j, \quad \text{where} \quad \overline{A}' = A, \quad \text{or} \quad \bar{a}_{ij} = a_{ji}.$$

The scalars of a quasi-field are the elements s such that $\bar{s} = s$. The diagonal elements of a hermitian matrix are therefore scalars. The process of completing squares is carried out in much the same way as in a field. Thus if, in f above, $a_{11} \neq 0$,

$$f = \left(\bar{x}_1 + \sum_{i=1}^{n} \bar{x}_i a_{i1} a_{11}^{-1}\right) a_{11} \left(x_1 + \sum_{i=2}^{n} a_{11}^{-1} a_{1i} x_i\right)$$
$$+ \sum_{j,k=2}^{n} \bar{x}_j (a_{jk} - a_{j1} a_{11}^{-1} a_{1k}) x_k.$$

Hence the analogue of a form like f_1 in Witt's theorem can be written

$$\bar{x}_1 a x_1 + \phi, \quad \text{where} \quad \phi = \sum_{i,j=2}^{n} \bar{x}_i b_{ij} x_j, \qquad \bar{b}_{ij} = b_{ji}.$$

Since determinants do not exist in a quasi-field (except for hermitian matrices), we cannot demonstrate that a matrix T is nonsingular

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[1] See bibliography.

by the nonvanishing of a determinant. Instead, we may construct explicitly the reciprocal matrix V, such that $VT=TV=I$.

We shall, in §3, consider automorphs of f, that is matrices T such that $\overline{T}'AT=A$. If z denotes the first column of T, then $\bar{z}'Az=a_{11}$, that is z is a representation of the leading coefficient a_{11} of f. If $a_{11}\neq 0$, we shall for any given representation z of a_{11} construct a corresponding automorph of f.

2. A generalization of Witt's theorem. The theorem we shall prove is the following.

THEOREM 2. *Let F be a quasi-field with a conjugate operation as described above, $2\neq 0$. Let a be a nonzero scalar, and B_1, B_2 nonsingular hermitian matrices of order $n-1$, with elements in F. Let*

$$A_1 = \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix}, \tag{1}$$

where 0 denotes a column, and $0'$ a row, of $n-1$ zeros. Let T denote any transformation (with coefficients in F) of A_1 into A_2, that is let

$$A_2 = \overline{T}'A_1T. \tag{2}$$

Then we can construct a transformation of B_1 into B_2.

PROOF. We can write (2) in the form

$$\begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} \bar{x}_0 & \bar{x}' \\ \bar{y} & \overline{T}_1' \end{bmatrix}\begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix}\begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix}, \tag{3}$$

where x_0 is a constant, x and y are column vectors of $n-1$ components, T_1 a matrix of order $n-1$. Expanding (3) we get

$$\bar{x}_0ax_0 + \bar{x}'B_1x = a, \tag{4}$$

$$\bar{x}_0ay' + \bar{x}'B_1T_1 = 0', \qquad \bar{y}ax_0 + \overline{T}_1'B_1x = 0, \tag{5}$$

$$\bar{y}ay' + \overline{T}_1'B_1T_1 = B_2. \tag{6}$$

Our problem is to derive from (4)–(6) a transformation of B_1 into B_2.

Suppose we could secure $x_0=1$, $x=0$, to begin with. Then by (5), $y'=0'$, $\bar{y}=0$; and by (6), $\overline{T}_1'B_1T_1=B_2$. What we shall do is construct a nonsingular automorph U of A_1 whose first column is the same as that of T. Having done this, let

$$W = \begin{bmatrix} z_0 & w' \\ z & U_1 \end{bmatrix} \text{ be the reciprocal of } U = \begin{bmatrix} x_0 & \cdot \\ x & \cdot \end{bmatrix}.$$

Then $z_0x_0+w'x=1$, $zx_0+U_1x=0$, and so

$$WT = \begin{bmatrix} z_0 & w' \\ z & U_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & u' \\ 0 & V \end{bmatrix},$$

say; and WT also replaces A_1 by A_2. By the preceding remark, $\bar{V}'B_1V = B_2$.

Hence Theorem 2 is a consequence of the following theorem; or see §4.

3. **Automorphs with an assigned first column.** We now prove:

THEOREM 3. *Let $A = (a_{ij})$ be any nonsingular hermitian matrix with coefficients in a quasi-field F of characteristic not 2. Let z be a representation in F of a_{11}, that is $\bar{z}'Az = a_{11}$, and assume $a_{11} \neq 0$. Then there exists in F a nonsingular automorph of A with z as its first column.*

We first complete squares, which amounts to applying a transformation

$$\bar{P}'AP = \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix} = A_1 \text{ say, where } P = \begin{bmatrix} 1 & v' \\ 0 & I \end{bmatrix}, \tag{7}$$

$$P^{-1} = \begin{bmatrix} 1 & -v' \\ 0 & I \end{bmatrix}.$$

Here $a = a_{11}$, and B_1 is hermitian with A. We have $\bar{z}'Az = \bar{u}'A_1u$, where $z = Pu$; and if we can construct an automorph U of A_1 with u as first column, then PUP^{-1} will be an automorph of A with z as first column. For, the first column of PU is Pu, and multiplication on the right by P^{-1} does not change the first column of PU.

We can therefore use the notations (4)–(6) with $B_2 = B_1$. Here x_0 and x are given as satisfying (4), and it is required to find y and T_1 to satisfy (5) and (6).

The cases $x_0 = 0$ and $x_0 \neq 0$ must be distinguished.

Let $x_0 = 0$. Then $\bar{x}'B_1x = a$, and we must choose y and T_1 to satisfy

$$\bar{x}'B_1T_1 = 0', \qquad \bar{y}ay' + \bar{T}_1'B_1T_1 = B_1. \tag{8}$$

The last equation can be replaced by $(\overline{xy'})'B_1(xy') + \bar{T}_1'B_1T_1 = B_1$, hence by

$$(\overline{xy' + T_1})'B_1(xy' + T_1) = B_1, \tag{9}$$

in view of (8_1). Then all of (8) will hold if we put

$$T_1 = I - xy', \qquad \bar{x}'B_1 - ay' = 0; \tag{10}$$

the last is satisfied if $y' = a^{-1}\bar{x}'B_1$. It will be found that $y'x = 1$, and

$$\begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} \begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{11}$$

Let $x_0 \neq 0$. Then (5) will hold if

$$\bar{y} = -\bar{T}_1' B_1 x x_0^{-1} a^{-1}, \qquad a y' = -\bar{x}_0^{-1} \bar{x}' B_1 T_1, \tag{12}$$

and then (6) becomes

$$\bar{T}_1'(B_1 + B_1 x (\bar{x}_0 a x_0)^{-1} \bar{x}' B_1) T_1 = B_1. \tag{13}$$

We therefore try to choose a number k in F to satisfy

$$(I + B_1 x k \bar{x}') B_1 (I + x \bar{k} \bar{x}' B_1) = B_1 + B_1 x (\bar{x}_0 a x_0)^{-1} \bar{x}' B_1. \tag{14}$$

In virtue of (4) this will be satisfied if

$$k + \bar{k} + k(a - \bar{x}_0 a x_0)\bar{k} = (\bar{x}_0 a x_0)^{-1}. \tag{15}$$

Here we try the substitution $k = t^{-1}$, and multiply left and right by t and $\bar{t}$ to obtain

$$\bar{t} + t + a - \bar{x}_0 a x_0 = t(\bar{x}_0 a x_0)^{-1} \bar{t}. \tag{16}$$

To satisfy this we put $t = h + \bar{x}_0 a x_0$, and find

$$h(\bar{x}_0 a x_0)^{-1} \bar{h} = a, \text{ which holds if } h = \pm a x_0, \quad t = (\bar{x}_0 \pm 1) a x_0. \tag{17}$$

Since $2 \neq 0$ in F we can choose the sign to make $t \neq 0$, so that k exists.

We can now solve for T_1 the equations

$$(I + x \bar{k} \bar{x}' B_1) T_1 = I = T_1 (I + x \bar{k} \bar{x}' B_1). \tag{18}$$

For if we put $T_1 = I + x m \bar{x}' B_1$, where m is a constant to be determined, then (18) will hold if

$$\bar{k} + m + \bar{k}(a - \bar{x}_0 a x_0) m = 0 = m + \bar{k} + m(a - \bar{x}_0 a x_0) \bar{k}. \tag{19}$$

Noting that $\bar{k}\bar{t} = \bar{t}\bar{k} = 1$ (since $k = t^{-1}$), we replace (19) by

$$m^{-1} + \bar{t} + a - \bar{x}_0 a x_0 = 0 = \bar{t} + m^{-1} + a - \bar{x}_0 a x_0, \tag{20}$$

which holds if $m^{-1} = -\bar{t} - a + \bar{x}_0 a x_0 = -(1 \pm \bar{x}_0) a$. Thus m exists, and

$$T_1 = I + x m \bar{x}' B_1. \tag{21}$$

Finally we verify that the automorph so constructed is nonsingular. By (3),

$$\begin{bmatrix} a^{-1}\bar{x}_0 a & a^{-1}\bar{x}' B_1 \\ B_1^{-1}\bar{y} a & B_1^{-1}\bar{T}_1' B_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{22}$$

Hence we have only to prove that[2]

[2] The referee remarked that (23) need not be proved since it is known that if $(\alpha)(\beta) = 1$, where (α), (β) are matrices with elements in a division ring, then also $(\beta)(\alpha) = 1$.

$$\begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} \begin{bmatrix} a^{-1}\bar{x}_0 a & a^{-1}\bar{x}'B_1 \\ B_1^{-1}\bar{y}a & B_1^{-1}\overline{T}_1'B_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{23}$$

Put $r=\bar{x}_0 a x_0$. Using $\bar{x}'B_1x=a-r$, and verifying that $m+\bar{m}+m(a-r)\bar{m}=-a^{-1}$, the proof of (23) is easy. For example, $x_0a^{-1}\bar{x}_0a+y'B_1^{-1}\bar{y}a=1$ if and only if $ra^{-1}r+\bar{x}'B_1(I+xm\bar{x}'B_1)B_1^{-1}(I+B_1x\bar{m}\bar{x}')B_1x=r$, or $ra^{-1}r+a-r+(a-r)(-a^{-1})(a-r)=r$, or $r=r$.

One additional remark is worth making for the case where F is a (commutative) field. If K is any square matrix of rank 1, it can of course be expressed as xy', where x and y are column vectors. The determinant of $I+K$ is readily found, since as is easily seen,

$$|I + xy'| = 1 + x'y = 1 + y'x = 1 + \sum x_iy_i.$$

The reciprocal of a matrix of the type $I+hxy'$ (where h is a constant) can be found by noting that

$$(I + hxy')(I + kxy') = I + \{h + k + hky'x\}xy',$$

and choosing k to make $h+k+hky'x=0$.

4. **An alternative construction for Theorem 2.** For some purposes, it is more advantageous not to construct an automorph of A_1, but to continue the argument from (4)–(6) as follows. If $x_0=0$, $xy'+T_1$ replaces B_1 by B_2. Let $x_0\neq 0$. Then (5) is equivalent to (12), and (6) reduces to (13) with B_2 instead of B_1 on the right. We have (14)–(17) as before, and so $T_1+x\bar{k}\bar{x}'B_1T_1$ is a transformation replacing B_1 by B_2. This transformation may in certain cases be integral (in a sense which we need not discuss here) even though no integral automorph of A_1 exists with x_0 and x as first column.

It should be mentioned that the preceding methods can be extended to the case where the element a is replaced by a nonsingular hermitian matrix of order greater than 1.

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