THE COEFFICIENTS OF UNIVALENT FUNCTIONS

M. S. ROBERTSON

1. Introduction. Let the function

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots, \qquad c_n \text{ real}, \tag{1.1}$$

be regular and convex in the direction of the imaginary axis for $|z| < 1$. Thus each circle $|z| = r$, $0 < r < 1$, is mapped by $f(z)$ into a contour C_r which has the property that straight lines parallel to the imaginary axis cut C_r in at most two points. Since the coefficients are all real, C_r is symmetric about the real axis. For

$$f(re^{i\theta}) = U(r, \theta) + iV(r, \theta)$$

we have $\partial U(r, \theta)/\partial\theta \leqq 0$ for $0 < \theta < \pi$. In other words, $zf'(z)$ is typically real for $|z| < 1$. It is well known [1, 2][1] that the coefficients c_n are bounded, $|c_n| \leqq |c_1|$, $n = 1, 2, \cdots$, and [3] have the representation

$$c_n = \frac{c_1}{n\pi} \int_0^\pi \frac{\sin n\theta}{\sin \theta} \, d\alpha(\theta) \tag{1.2}$$

where $\alpha(\theta)$ is a nondecreasing function of θ in $(0, \pi)$ normalized so that

$$\frac{1}{\pi} \int_0^\pi d\alpha(\theta) = 1.$$

A sufficient condition that $f(z)$, given by the series (1.1), be regular and convex in the direction of the imaginary axis for $|z| < 1$ is that the sequence $\{c_n\}$ be monotonic of order 4, a theorem due to L. Fejér [4]. A sequence $\{c_n\}$ is said to be monotonic of order p if each of the differences

$$\Delta^{(k)} c_n = c_n - C_{k,1} c_{n+1} + C_{k,2} c_{n+2} - \cdots + (-1)^k C_{k,k} c_{n+k} \tag{1.3}$$

are non-negative for $k = 0, 1, 2, \cdots, p$; $n = 0, 1, 2, \cdots$. This sufficiency test implies, among other inequalities, that $0 \leqq c_n - c_{n+1}$. This suggests the problem of finding an upper bound for $c_n - c_{n+1}$ for functions $f(z)$ given by (1.1) which are convex in the direction of the imaginary axis for $|z| < 1$. The example $c_1 z(1+z)^{-1}$ shows that the upper bound $2|c_1|$ is sharp. However, if we consider the differences $c_{n-1} - c_{n+1}$ we obtain an inequality which is not so immediately obvious. This inequality is stated in the following theorem.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM A. *Let the function*

$$f(z) = z + c_2z^2 + \cdots + c_nz^n + \cdots$$

be regular and convex in the direction of the imaginary axis for $|z| < 1$ *and be real on the real axis. Then the coefficients* c_n *satisfy the inequalities*

$$c_{n-1} - c_{n+1} \leqq 4n(n^2 - 1)^{-1}(1 - |c_n|), \qquad n = 2, 4, 6, \cdots; c_1 = 1;$$
$$|c_{n-1} - c_{n+1}| \leqq 4n(n^2 - 1)^{-1}(1 - c_n), \qquad n = 3, 5, 7, \cdots.$$

The factor $4n(n^2-1)^{-1}$ *cannot be replaced by a smaller one and the equality signs are attained for the function* $z(1-z)^{-1}$.

2. **Some trigonometric inequalities.** For the proof of Theorem A we need the trigonometric inequality to follow.

LEMMA 1. *If* n *is a positive integer then for all real values of* θ

$$(2.1) \qquad (n+1)\frac{\sin (n-1)\theta}{\sin \theta} - (n-1)\frac{\sin (n+1)\theta}{\sin \theta} \leqq 4\left(n - \frac{\sin n\theta}{\sin \theta}\right).$$

PROOF. We consider first the case where n is an odd positive integer.

$$8\sum_{k=1}^{(n-1)/2}\left[\cos\frac{n\theta}{2} - \cos (2k-1)\frac{\theta}{2}\right]^2$$
$$= (4n-4)\cos^2\frac{n\theta}{2} - 16\cos\frac{n\theta}{2}\cdot\sum_{1}^{(n-1)/2}\cos (2k-1)\frac{\theta}{2} + 8\sum_{1}^{(n-1)/2}\cos^2 (2k-1)\frac{\theta}{2}$$
$$= (4n-4)\cos^2\frac{n\theta}{2} - 8\cos\frac{n\theta}{2}\cdot\frac{\sin ((n-1)\theta/2)}{\sin (\theta/2)} + 8\left[\frac{n-1}{4} + \frac{\sin (n-1)\theta}{4\sin\theta}\right]$$
$$= (2n-2)(1+\cos n\theta) - 4\left[\frac{\sin ((2n-1)\theta/2)}{\sin (\theta/2)} - 1\right] + 2n - 2 + 2\frac{\sin (n-1)\theta}{\sin\theta}$$

$$= 4n + (2n - 2)\cos n\theta + 4\left[\frac{\sin (n-1)\theta}{\sin \theta} - \frac{\sin ((2n-1)\theta/2)}{\sin (\theta/2)}\right]$$

$$- 2\,\frac{\sin (n-1)\theta}{\sin \theta}$$

$$= 4n + (2n - 2)\cos n\theta - 4\,\frac{\sin n\theta}{\sin \theta} - 2\,\frac{\sin (n-1)\theta}{\sin \theta}$$

$$= 4\left(n - \frac{\sin n\theta}{\sin \theta}\right) - (n+1)\,\frac{\sin (n-1)\theta}{\sin \theta} + (n-1)\,\frac{\sin (n+1)\theta}{\sin \theta}.$$

Again, if n is an even integer we have

$$4\left(1 - \cos\frac{n\theta}{2}\right)^2 + 8\sum_{k=1}^{(n-2)/2}\left[\cos\frac{n\theta}{2} - \cos k\theta\right]^2$$

$$= \left(4 - 8\cos\frac{n\theta}{2} + 4\cos^2\frac{n\theta}{2}\right) + (4n - 8)\cos^2\frac{n\theta}{2}$$

$$- 16\cos\frac{n\theta}{2}\sum_{1}^{(n-2)/2}\cos k\theta + 8\sum_{1}^{(n-2)/2}\cos^2 k\theta$$

$$= 4 - 8\cos\frac{n\theta}{2} + 2(1 + \cos n\theta) + (2n - 4)(1 + \cos n\theta)$$

$$- 16\cos\frac{n\theta}{2}\left[-\frac{1}{2} + \frac{\sin ((n-1)\theta/2)}{2\sin (\theta/2)}\right]$$

$$+ 8\left[\frac{n-3}{4} + \frac{\sin (n-1)\theta}{4\sin \theta}\right]$$

$$= 4n - 4 + (2n - 2)\cos n\theta - 8\cos\frac{n\theta}{2}\cdot\frac{\sin ((n-1)\theta/2)}{\sin (\theta/2)}$$

$$+ 2\,\frac{\sin (n-1)\theta}{\sin \theta}$$

$$= 4\left(n - \frac{\sin n\theta}{\sin \theta}\right) - (n+1)\,\frac{\sin (n-1)\theta}{\sin \theta} + (n-1)\,\frac{\sin (n+1)\theta}{\sin \theta}.$$

Hence, in either case we have, as required,

$$\begin{aligned} &4\left(n - \frac{\sin n\theta}{\sin \theta}\right) + (n-1)\,\frac{\sin (n+1)\theta}{\sin \theta} \\ &\qquad - (n+1)\,\frac{\sin (n-1)\theta}{\sin \theta} \geqq 0. \end{aligned} \tag{2.2}$$

LEMMA 2. *If n is a positive integer then for all real values of θ*

$$(2.3)\qquad 4\left[1+\frac{\sin 2\theta}{\sin\theta}+\frac{\sin 3\theta}{\sin\theta}+\cdots+\frac{\sin (n-1)\theta}{\sin\theta}\right] - (n-2)\frac{\sin n\theta}{\sin\theta} \leqq n^2.$$

Lemma 2 follows easily from the identity

$$(2.4)\qquad \frac{\sin n\theta}{\sin\theta}+2\sum_{k=1}^{n-1}\frac{\sin k\theta}{\sin\theta}=\left(\frac{\sin (n\theta/2)}{\sin (\theta/2)}\right)^2$$

and the inequality $A_n(0)\geqq 0$ where

$$(2.5)\qquad A_n(\theta)\equiv n^2+n\frac{\sin n\theta}{\sin\theta}-2\left(\frac{\sin (n\theta/2)}{\sin (\theta/2)}\right)^2.$$

To prove the inequality $A_n(\theta)\geqq 0$ we observe first that $A_1(\theta)=A_2(\theta)\equiv 0$. We prove next that $A_{n+1}(\theta)\geqq A_{n-1}(\theta)$ whence $A_n(\theta)\geqq 0$ by induction.

$$(2.6)\qquad \begin{aligned} A_{n+1}(\theta) &- A_{n-1}(\theta) \\ &= 4n+(n+1)\frac{\sin (n+1)\theta}{\sin\theta}-(n-1)\frac{\sin (n-1)\theta}{\sin\theta} \\ &\quad -2\left[\left(\frac{\sin ((n+1)\theta/2)}{\sin (\theta/2)}\right)^2-\left(\frac{\sin ((n-1)\theta/2)}{\sin (\theta/2)}\right)^2\right] \\ &= 4n+(n-1)\left[\frac{\sin (n+1)\theta-\sin (n-1)\theta}{\sin\theta}\right] \\ &\quad +2\frac{\sin (n+1)\theta}{\sin\theta}-4\frac{\sin n\theta}{\sin\theta}(1+\cos\theta) \\ &= 4\left(n-\frac{\sin n\theta}{\sin\theta}\right)+(n-1)\frac{\sin (n+1)\theta}{\sin\theta} \\ &\quad -(n+1)\frac{\sin (n-1)\theta}{\sin\theta}\geqq 0 \end{aligned}$$

by Lemma 1.

3. **Proof of Theorem A.** From (2.1) we have

$$(3.1)\qquad \frac{1}{(n-1)\pi}\int_0^{\pi}\frac{\sin (n-1)\theta}{\sin\theta}\,d\alpha(\zeta)$$

$$
\begin{aligned}
&-\frac{1}{(n+1)\pi}\int_0^{\pi}\frac{\sin(n+1)\theta}{\sin\theta}\,d\alpha(\theta)\\
&\leqq\frac{4n}{n^2-1}\left[\frac{1}{\pi}\int_0^{\pi}d\alpha(\theta)-\frac{1}{n\pi}\int_0^{\pi}\frac{\sin n\theta}{\sin\theta}\,d\alpha(\theta)\right].
\end{aligned}
\tag{3.1}
$$

From (1.2) and (3.1) we have immediately

$$
c_{n-1}-c_{n+1}\leqq\frac{4n}{n^2-1}(1-c_n),\qquad n=2,3,\cdots. \tag{3.2}
$$

Since

$$
-f(-z)=z-c_2z^2+c_3z^3-c_4z^4\cdots \tag{3.3}
$$

is also regular and convex in the direction of the imaginary axis for $|z|<1$ it follows that (3.2) may be replaced by the inequalities given in Theorem A.

The function

$$
f(z)=\sum_1^{\infty}\frac{1}{n}\frac{\sin n\theta}{\sin\theta}z^n \tag{3.4}
$$

is regular and convex in the direction of the imaginary axis for $|z|<1$ since $zf'(z)=z(1-2z\cos\theta+z^2)^{-1}$ is typically real, indeed univalent and star-like for $|z|<1$. For this function

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{c_{n-1}-c_{n+1}}{1-c_n}
&=\lim_{\theta\to 0}\left(\frac{n}{n^2-1}\right)\frac{(n+1)\sin(n-1)\theta-(n-1)\sin(n+1)\theta}{n\sin\theta-\sin n\theta}\\
&=4n\cdot(n^2-1)^{-1}.
\end{aligned}
$$

As a consequence of Theorem A we easily obtain the following theorem.

THEOREM B. *Let the function*

$$
F(z)=z+a_2z^2+\cdots+a_nz^n+\cdots
$$

have real coefficients and be regular and univalent for $|z|<1$. *Then*

$$
\begin{aligned}
(n+1)a_{n-1}-(n-1)a_{n+1}&\leqq 4(n-|a_n|),\quad n=2,4,6,\cdots,\\
|(n+1)a_{n-1}-(n-1)a_{n+1}|&\leqq 4(n-a_n),\qquad n=3,5,7,\cdots.
\end{aligned}
\tag{3.5}
$$

The equality signs are attained by the function $z\cdot(1-z)^{-2}$ *which is univalent in the unit circle.*

4. Some additional inequalities for the coefficients. From the inequality (2.3) by an argument similar to that used in the proof of Theorems A and B we obtain the following theorem.

THEOREM C. *With the same hypothesis as in Theorem* B, *the following inequalities for the coefficients of* $F(z)$ *hold*:

$$4(1 + a_2 + a_3 + \cdots + a_{n-1}) - (n - 2)a_n \leqq n^2. \tag{4.1}$$

In a somewhat similar way by using the trigonometric inequality

$$\begin{aligned} &(n^2 - 1)(n + 2)\left(2 - \frac{\sin 2\theta}{\sin\theta}\right) - 2(n - 1)(n + 2)\frac{\sin (n + 1)\theta}{\sin\theta} \\ &\quad + (n + 3)(n + 2)\frac{\sin n\theta}{\sin\theta} + (n^2 - 3n - 2)\frac{\sin (n + 2)\theta}{\sin\theta} \geqq 0 \end{aligned} \tag{4.2}$$

we may obtain another set of inequalities for the coefficients of $f(z)$ of Theorem A:

$$2(n^2 - 1)(1 - c_2 - c_{n+1}) + (n^2 + 3n)c_n + (n^2 - 3n - 2)c_{n+2} \geqq 0. \tag{4.3}$$

We omit the details of the proof.

REFERENCES

1. M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. vol. 37 (1936) pp. 374–408.

2. ———, *Analytic functions star-like in one direction*, Amer. J. Math. vol. 58 (1936) pp. 465–472.

3. ———, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 565–572.

4. L. Fejér, *Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 18–59.

RUTGERS UNIVERSITY