If $p=q+1$, replace (21) by

$$\begin{aligned}[(2\alpha_k - \beta_q)(1-x) + (A-B)x]F \\ = \alpha_k(1-x)F(\alpha_k+) + (\alpha_k-\beta_q)F(\alpha_k-) \\ - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j+); \qquad k=1,2,\cdots,p.\end{aligned} \tag{30}$$

University of Michigan

ON THE GROWTH OF THE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. ROSENBLATT

In a recent paper,[1] N. Levinson gave four theorems concerning the behaviour of the solutions of the differential equation of elastic vibrations

$$d^2x/dt^2 + \phi(t)x = 0 \tag{1}$$

as $t\rightarrow+\infty$. It is the purpose of this note to give generalizations of the Theorems I and III of Levinson by making use of certain inequalities concerning homogeneous equations of the first order

$$\frac{dx_i}{dt} + \sum_{k=1}^{n} a_{ik}x_k = 0, \qquad i=1,\cdots,n. \tag{2}$$

Theorems I and III of Levinson run as follows:

Theorem I. *If $\alpha(t)$ denotes the integral*

$$\alpha(t) = \int_0^t |\phi(t) - c^2|\,dt, \tag{3}$$

then

$$x(t) = O\{\exp(\alpha(t)/2c)\}. \tag{4}$$

Theorem III. *If $\alpha(t)$ is $O(t)$ then*

$$\limsup_{t\rightarrow\infty} |x(t)\exp(\alpha(t)/2c)| > 0. \tag{5}$$

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[1] *The growth of the solutions of a differential equation*, Duke Math. J. vol. 8 (1941) pp. 1–11.

The Polish mathematician Z. Butlewski in a paper written in Polish[2] sets

$$r = \left(\sum_1^n x_i^2\right)^{1/2}, \tag{6}$$

$$\phi(t) = \sum_1^n a_{ii}\left(\frac{x_i}{r}\right)^2 + \sum_{i,k=1}^n (a_{ik} + a_{ki}) \frac{x_i x_k}{r^2}, \tag{7}$$

where $i \neq k$, $i < k$, and obtains immediately from (2)

$$r = C \exp\left(-\int_{t_0}^t \phi(\tau) d\tau\right). \tag{8}$$

Setting

$$\alpha_{ii} = \int_{t_0}^t |a_{ii}| d\tau, \qquad \beta_{ij} = \int_{t_0}^t |a_{ij} + a_{ji}| d\tau, \tag{9}$$

we have

$$r \leqq C \exp\left(\sum_1^n \alpha_{ii} + \frac{1}{2} \sum_{i,\, j=1;\, i<j}^n \beta_{ij}\right) \tag{10}$$

and from this we have the following theorem.

THEOREM XVIII OF BUTLEWSKI. *If $\alpha_{ii} < +\infty$ and $\beta_{ij} < +\infty$, all the systems x_i, $i = 1, \cdots, n$, of solutions of* (2) *are bounded.*

With the designation

$$M(t) = \max \int_{t_0}^t |a_{ij}| d\tau, \tag{11}$$

we have

$$r \leqq C \exp(nM(t)). \tag{12}$$

We can complete Butlewski's theorem by remarking that

$$r \geqq C \exp\left(-\sum_1^n \alpha_{ii} - \frac{1}{2} \sum_{i,\, j=1;\, i<j}^n \beta_{ij}\right), \tag{13}$$

and

$$r \geqq C \exp(-nM(t)). \tag{14}$$

[2] O całkach rzeczywistych równań różniczkowych zwyczajnych, Wiadomości Matematyczne vol. 44 (1937) pp. 17–81.

In the particular case $n=2$ Butlewski introduces polar coordinates

$$x_1 = \rho \cos \phi, \qquad x_2 = \rho \sin \phi \tag{15}$$

and obtains

$$\rho = C \exp\left(- \int_{t_0}^{t} \{a_{11} \cos^2 \phi + (a_{12} + a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi)\} d\tau \right). \tag{16}$$

The maximum of $F(\phi) = -\{a_{11} \cos^2 \phi + (a_{12}+a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi\}$ is

$$2^{-1}\{- a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\},$$

so that

$$\rho \leqq C \exp\left(\frac{1}{2} \int_{t_0}^{t} \{- a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau \right). \tag{17}$$

Thus we obtain the following theorem.

THEOREM XIX OF BUTLEWSKI. *x_1, x_2 are limited if*

$$\frac{1}{2} \int_{t_0}^{t} \{- a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau) \tag{18}$$

is limited.

In the same manner we obtain

$$\rho \geqq C \exp\left(\frac{1}{2} \int_{t_0}^{t} \{- a_{11} - a_{22} - ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau \right), \tag{19}$$

completing Butlewski's results.

The linear differential equation

$$x'' + \psi(t) x' + \phi(t) x = 0 \tag{20}$$

with continuous ϕ and ψ can be transformed into a system (2) by setting

$$x' = \lambda x_2, \qquad x = x_1, \tag{21}$$

$\lambda \neq 0$, continuous in t. We obtain the system

$$(22) \qquad \frac{dx_1}{dt} - \lambda x_2 = 0, \qquad \frac{dx_2}{dt} + \frac{\phi}{\lambda} x_1 + \left(\frac{\lambda'}{\lambda} + \psi\right) x_2 = 0,$$

$a_{11}=0$, $a_{12}=-\lambda$, $a_{21}=\phi/\lambda$, $a_{22}=\lambda'/\lambda+\psi$.

The inequalities (17), (19) now take the form

$$(23) \qquad \rho \leqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{\lambda'}{\lambda} - \psi + \left(\left(\frac{\lambda'}{\lambda}+\psi\right)^2 + \left(\frac{\phi}{\lambda}-\lambda\right)\right)^{1/2}\right\} d\tau\right),$$

$$(24) \qquad \rho \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{\lambda'}{\lambda} - \psi - \left(\left(\frac{\lambda'}{\lambda}+\psi\right)^2 + \left(\frac{\phi}{\lambda}-\lambda\right)^2\right)^{1/2}\right\} d\tau\right).$$

Taking $\lambda = c$, Butlewski obtains

$$(25) \qquad \rho \leqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi + \left(\psi^2 + \left(\frac{\phi}{c} - c\right)^2\right)^{1/2}\right\} d\tau\right)$$

and we can add the inequality

$$(26) \qquad \rho \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi - \left(\psi^2 + \left(\frac{\phi}{c} - c\right)^2\right)^{1/2}\right\} d\tau\right).$$

We shall now generalize Levinson's Theorem III for the equation (20).

We can suppose $x>0$, $x'<0$, otherwise we should have an infinite number of values $t=t_i$, $i=1, 2, \cdots ; t_i \to \infty$, with

$$(27) \qquad |x(t_i)| \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi - \left(\psi^2 + \left(\frac{\phi}{c} - c\right)^2\right)^{1/2}\right\} d\tau\right).$$

Consider the intervals $n \leqq t \leqq n+1$. We have

$$x(n+1) - x(n) = \int_n^{n+1} x'(t)dt,$$

and denoting by x_n' the maximum of $x'(t)$ in $\langle n, n+1\rangle$,

$$x(n) \geqq -x_n', \qquad x_n' = x'(t_n), \qquad n \leqq t_n \leqq n+1.$$

We have

$$x'(n) - x_n' = \int_{t_n}^{n} x''(t)dt = -\int_{t_n}^{n} (\phi x + \psi x')dt,$$

$$| x'(n) | \leqq x(n) + x(n) \int_{n}^{n+1} | \phi | dt + | \psi |_{\max} x(n)$$

$$= x(n) \left\{ 1 + \int_{n}^{n+1} | \phi | dt + | \psi |_{\max} \right\}.$$

$|\psi|_{\max}$ is the maximum of $|\psi|$ in $n \leqq t \leqq n+1$.

We obtain

$$(28) \qquad x(n) \geqq \frac{C \exp\left(\frac{1}{2}\int_{t_0}^{t} \left\{ -\psi - \left(\psi^2 + \left(\frac{\phi}{c} - c\right)^2\right)^{1/2} \right\} d\tau\right)}{1 + \frac{1}{c^2}\left(1 + |\psi|_{\max} + \int_{n}^{n+1} |\phi| dt\right)}.$$

The result is the following theorem.

THEOREM. *There exist infinite values of* $t = t_i$, $t_i \to \infty$, *for which we have* (27) *if the following conditions are satisfied*:

1. $\alpha(t)$ *is of order* $O(t)$.
2. $|\psi|$ *is bounded.*

UNIVERSITY OF SAN MARCOS