THE CONTIGUOUS FUNCTION RELATIONS FOR ${}_pF_q$ WITH APPLICATIONS TO BATEMAN'S $J_n^{u,v}$ AND RICE'S $H_n(\zeta, p, v)$

EARL D. RAINVILLE

1. Introduction. If in the generalized[1] hypergeometric function

$$
\begin{aligned}
(1)\qquad {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{x^n}{n!}; \\
(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), &\ (\alpha)_0 = 1,
\end{aligned}
$$

one, and only one, of the parameters is increased or decreased by unity, the resultant function is said to be contiguous to the ${}_pF_q$ in (1). We restrict ourselves to the case $p \leqq q+1$, in order to insure a nonzero radius of convergence for the series in (1), and note that no β is permitted to be either zero or a negative integer.

For the ordinary hypergometric function ${}_2F_1$ Gauss[2] obtained fifteen relations each expressing ${}_2F_1$ linearly in terms of two of its six contiguous functions and with coefficients polynomials at most linear in x. Instead of the fifteen relations, it is often convenient to use a set of five linearly independent ones chosen from among them. The other relations all follow from the basic set.

Throughout this study the parameters stay fixed and the work is concerned only with the function ${}_pF_q$ and its contiguous functions. Hence, we are able to use an abbreviated notation illustrated by the following:

$$
\begin{aligned}
F &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\alpha_1+) &= {}_pF_q(\alpha_1+1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\beta_1-) &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1-1, \beta_2, \cdots, \beta_q; x).
\end{aligned}
$$

There are, of course, $(2p+2q)$ functions contiguous to F. Corresponding to Gauss' five independent relations in the case of ${}_2F_1$ there is for F a set of $(2p+q)$ linearly independent relations, which we shall obtain. The canonical form into which we put this basic set may be described as follows:

First, there are $(p+q-1)$ relations each containing F and two of its contiguous functions. These will be called the simple relations. Each

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[1] For an extensive treatment see W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tract No. 32, 1935.

[2] Gauss, *Werke*, vol. 3, p. 130.

simple relation connects F, $F(\alpha_1+)$, and $F(\alpha_k+)$ for $k=2, 3, \cdots, p$, or it connects F, $F(\alpha_1+)$, and $F(\beta_j-)$ for $j=1, 2, \cdots, q$. The simple relations are immediate extensions of two of Gauss' five relations and are not novel in any way.

Second, there are $(p+1)$ less simple relations each containing F and $(q+1)$ of its contiguous functions. In our canonical form we shall select these so that one of them connects F, $F(\alpha_1+)$, and all the functions $F(\beta_j+)$ for $j=1, 2, \cdots, q$. Each of the other p relations will contain F, all the functions $F(\beta_j+)$; $j=1, 2, \cdots, q$, and one of the functions $F(\alpha_k-)$ for $k=1, 2, \cdots, p$. The less simple relations are generalizations of three of Gauss' five relations but differ from them in one essential aspect in that each relation contains F and $(q+1)$ of its contiguous functions. For Gauss' case $q+1=2$ and the less simple relations contain the same number of contiguous functions as do the simple ones.

Since we shall actually exhibit the $(2p+q)$ relations, it will be evident upon looking at them that, just as in the case of the ordinary hypergeometric function, the coefficients are polynomials at most linear in x.

As examples of the use of the contiguous function relations, we shall obtain recurrence relations (not all previously known) for Bateman's[3] $J_n^{u,v}$ and for Rice's[4] $H_n(\zeta, p, v)$.

2. **The number and type of relations.** It is not difficult to determine, by a standard procedure[5] used in the case of the ordinary hypergeometric function, precisely how many contiguous function relations to expect and how many contiguous functions should appear in each relation. That procedure appears, however, to be ill-adapted to the actual determination of the coefficients. Hence, granted in advance the number and type of relations to be obtained, we shall get them by a simpler process.

3. **Notations and preliminary formulas.** It is convenient to use the following notations:

$$\prod_{s=1,(k)}^{m} A_s = \prod_{s=1}^{k-1} A_s \cdot \prod_{s=k+1}^{m} A_s, \tag{2}$$

a symbol denoting a product with a particular factor deleted,

[3] H. Bateman, *Two systems of polynomials for the solution of Laplace's integral equation*, Duke Math. J. vol. 2 (1936) pp. 569–577.

[4] S. O. Rice, *Some properties of* ${}_3F_2(-n, n+1, \zeta; 1, p; v)$, Duke Math. J. vol. 6 (1940) pp. 108–119.

[5] E. G. C. Poole, *Linear differential equations*, Oxford, 1936, pp. 92–93.

$$U_j = \frac{\prod_{s=1}^{p} (\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q} (\beta_s - \beta_j)}, \tag{3}$$

$$W_{j,k} = \frac{\prod_{s=1,(k)}^{p} (\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q} (\beta_s - \beta_j)}, \tag{4}$$

$$c_n = \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n}, \tag{5}$$

$$S_n = \frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_p + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_q + n)}, \tag{6}$$

$$\tau_{n,k} = \frac{S_n}{\alpha_k + n}, \tag{7}$$

$$A = \sum_{s=1}^{p} \alpha_s, \tag{8}$$

$$B = \sum_{s=1}^{q} \beta_s. \tag{9}$$

An examination of (5) and (6) shows that

$$c_{n+1} = S_n c_n. \tag{10}$$

The relation $\alpha(\alpha+1)_n = (\alpha+n)(\alpha)_n$, together with the definitions of the contiguous functions, yields the formulas:

$$F = \sum_{n=0}^{\infty} \frac{c_n x^n}{n!},$$

$$F(\alpha_k +) = \sum_{n=0}^{\infty} \frac{\alpha_k + n}{\alpha_k} \frac{c_n x^n}{n!}, \quad F(\alpha_k -) = \sum_{n=0}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n - 1} \frac{c_n x^n}{n!}, \tag{11}$$

$$F(\beta_k +) = \sum_{n=0}^{\infty} \frac{\beta_k}{\beta_k + n} \frac{c_n x^n}{n!}, \quad F(\beta_k -) = \sum_{n=0}^{\infty} \frac{\beta_k + n - 1}{\beta_k - 1} \frac{c_n x^n}{n!}.$$

4. **The $(p+q-1)$ simple relations.** Using the operator $\theta = x(d/dx)$, we see that

$$(\theta + \alpha_k)F = \sum_{n=0}^{\infty} (\alpha_k + n) \frac{c_n x^n}{n!}.$$

Hence, with the aid of (11),

$$(\theta + \alpha_k)F = \alpha_k F(\alpha_k +); \qquad k = 1, 2, \cdots, p. \tag{12}$$

Similarly, it follows that

$$(\theta + \beta_k - 1)F = (\beta_k - 1)F(\beta_k -); \qquad k = 1, 2, \cdots, q. \tag{13}$$

The $(p+q)$ equations (12) and (13) lead at once by elimination of θF to $(p+q-1)$ linear algebraic relations between F and pairs of its contiguous functions. Let us use $F(\alpha_1+)$ as an element in each equation. The result is the set of simple relations,

$$(\alpha_1 - \alpha_k)F = \alpha_1 F(\alpha_1 +) - \alpha_k F(\alpha_k +); \qquad k = 2, 3, \cdots, p, \tag{14}$$

and

$$(\alpha_1 - \beta_k + 1)F = \alpha_1 F(\alpha_1 +) - (\beta_k - 1)F(\beta_k -); \qquad k = 1, 2, \cdots, q. \tag{15}$$

5. **A relation involving $(q+1)$ contiguous functions.** From

$$F = \sum_{n=0}^{\infty} \frac{c_n x^n}{n!},$$

it follows that

$$\theta F = \sum_{n=1}^{\infty} \frac{n c_n x^n}{n!} = x \sum_{n=0}^{\infty} \frac{c_{n+1} x^n}{n!}.$$

Thus, because of (10) and (6),

$$\theta F = x \sum_{n=0}^{\infty} S_n \frac{c_n x^n}{n!},$$

where

$$S_n = \frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_p + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_q + n)}.$$

Now, if $p<q$, then the degree of the numerator of S_n is lower than the degree of the denominator and the elementary theory of rational fraction expansions yields

$$S_n = \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}, \qquad p < q,$$

in which the U_j is as defined in (3).

Therefore,

$$\theta F = x \sum_{n=0}^{\infty} \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n} \frac{c_n x^n}{n!},$$

which in view of (11) becomes

$$\theta F = x \sum_{j=1}^{q} U_j F(\beta_j +). \tag{16}$$

The elimination of θF using (16) and the case $k=1$ of (12) leads to

$$\alpha_1 F = \alpha_1 F(\alpha_1 +) - x \sum_{j=1}^{q} U_j F(\beta_j +), \qquad p < q. \tag{17}$$

If $p=q$, the degree of the numerator of S_n equals that of the denominator. However, when $p=q$,

$$S_n = 1 + \frac{\prod_{s=1}^{p} (\alpha_s + n) - \prod_{s=1}^{q} (\beta_s + n)}{\prod_{s=1}^{q} (\beta_s + n)},$$

in which the fraction on the right has the desired property that its numerator is of lower degree than its denominator. Thus

$$S_n = 1 + \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}, \qquad p = q,$$

and it is easy to see that in this case

$$\theta F = xF + x \sum_{j=1}^{q} U_j F(\beta_j +),$$

so that (17) is replaced by the relation

$$(\alpha_1 + x)F = \alpha_1 F(\alpha_1 +) - x \sum_{j=1}^{q} U_j F(\beta_j +), \qquad p = q. \tag{18}$$

If $p=q+1$, then with the notation of (8) and (9) we may write

$$S_n = n + A - B + \frac{\prod_{s=1}^{p} (\alpha_s + n) - (n + A - B) \prod_{s=1}^{q} (\beta_s + n)}{\prod_{s=1}^{q} (\beta_s + n)},$$

in which the fraction on the right has the desired rational fraction expansion, so that

$$S_n = n + A - B + \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}.$$

Thus we conclude that, when $p=q+1$,

$$\theta F = x\theta F + (A - B)xF + x\sum_{j=1}^{q} U_j F(\beta_j +),$$

so that (17) is this time to be replaced by the relation

$$\begin{aligned}[(1 - x)\alpha_1 + (A - B)x]F &= (1 - x)\alpha_1 F(\alpha_1 +) \\ &\quad - x\sum_{j=1}^{q} U_j F(\beta_j +); \qquad p = q + 1.\end{aligned} \tag{19}$$

6. **The remaining p relations.** Since

$$(\alpha_k - 1)_n = \frac{(\alpha_k - 1)(\alpha_k)_n}{\alpha_k + n - 1},$$

it is possible to write

$$\theta F(\alpha_k -) = \sum_{n=1}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n - 1} \frac{nc_n x^n}{n!},$$

or

$$\theta F(\alpha_k -) = x\sum_{n=0}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n} \frac{c_{n+1} x^n}{n!}.$$

Now, with the notation of (7),

$$\frac{c_{n+1}}{\alpha_k + n} = c_n \tau_{n,k},$$

where $\tau_{n,k}$ has, for $p \leqq q$, its numerator of lower degree than its denominator. Thus

$$\theta F(\alpha_k -) = (\alpha_k - 1)x\sum_{n=0}^{\infty} \tau_{n,k} \frac{c_n x^n}{n!},$$

where

$$\tau_{n,k} = \sum_{j=1}^{q} \frac{\beta_j W_{j,k}}{\beta_j + n}$$

in which the $W_{j,k}$ is as defined in (4).

We now have

$$\theta F(\alpha_k -) = (\alpha_k - 1)x\sum_{j=1}^{q} W_{j,k} F(\beta_j +); \qquad p \leqq q;\ k = 1, 2, \cdots, p.$$

But, from (12),

$$\theta F(\alpha_k -) = (\alpha_k - 1)[F - F(\alpha_k -)].$$

The elimination of $\theta F(\alpha_k -)$ from the above two formulas yields the p relations:

$$(20) \qquad F = F(\alpha_k -) + x\sum_{j=1}^{q} W_{j,k}F(\beta_j +); \qquad p \leqq q;\ k = 1, 2, \cdots, p.$$

When $p = q+1$ the fraction $\tau_{n,k}$ has its numerator and denominator of equal degree. Then we write

$$\tau_{n,k} = 1 + \frac{\prod_{s=1,(k)}^{p} (\alpha_s + n) - \prod_{s=1}^{q} (\beta_s + n)}{\prod_{s=1}^{q} (\beta_s + n)}$$

and conclude in the same manner as before that

$$\theta F(\alpha_k -) = (\alpha_k - 1)xF + (\alpha_k - 1)x\sum_{j=1}^{q} W_{j,k}F(\beta_j +).$$

Therefore, for $p = q+1$, (20) is to be replaced by

$$(21) \qquad (1 - x)F = F(\alpha_k -) + x\sum_{j=1}^{q} W_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.$$

7. **Summary of contiguous function relations.** We have shown that for

$$F(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x)$$

in which no two β's are equal and no β is a nonpositive integer, a canonical set of $(2p+q)$ contiguous function relations is as described below.

If $p<q$, (14), (15), (17), and (20) hold, with

$$U_j = \frac{\prod_{s=1}^{p} (\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q} (\beta_s - \beta_j)},$$

and

$$W_{j,k} = \frac{U_j}{\alpha_k - \beta_j}.$$

If $p=q$, the relations are (14), (15), and (20) above together with (18) to replace (17).

If $p=q+1$, the relations are (14) and (15) above together with (19), in which

$$A = \sum_{s=1}^{p} \alpha_s, \qquad B = \sum_{s=1}^{q} \beta_s,$$

and (21).

8. **Application to Bateman's $J_n^{u,v}$.** Bateman[6] has discussed the polynomials $x^{-u}J_n^{u,v}$ defined as

$$\frac{\Gamma(v + u/2 + n + 1)}{n!\Gamma(u + 1)\Gamma(v + u/2 + 1)} F(-\,n;\, u + 1,\, v + u/2 + 1;\, x^2).$$

Since a ${}_1F_2$ is involved, we may write down four relations using (15), (17), and (20) above with $\alpha_1 = -n$, $\beta_1 = u+1$, $\beta_2 = v+u/2+1$ and with x replaced by x^2. These four relations may be translated into recurrence formulas for the $J_n^{u,v}$. When a redundancy is eliminated, the contiguous function relations are seen to be equivalent to the set

$$J_n^{u,v} = J_{n-1}^{u,v} + J_n^{u,v-1}, \qquad uJ_n^{u,v-1/2} = xJ_n^{u-1,v} + xJ_{n-1}^{u+1,v},$$

and

$$(n + u)J_n^{u,v} = (n + v + u/2)J_{n-1}^{u,v} + xJ_n^{u-1,v+1/2}. \tag{22}$$

Bateman gives the first two of these relations, but (22) seems to be new.

The $J_n^{u,v}$ satisfy a pure recurrence relation,

$$\begin{aligned} n(n + u)J_n^{u,v} = {} & [(n + v + u/2)(n + u) \\ & + (n - 1)(2n + v + 3u/2 - 1) - x^2]J_{n-1}^{u,v} \\ & - (n + v + u/2 - 1)(3n + v + 3u/2 - 3)J_{n-2}^{u,v} \\ & + (n + v + u/2 - 1)(n + v + u/2 - 2)J_{n-3}^{u,v}, \end{aligned} \tag{23}$$

which may be obtained from the three relations given above it or, more easily, directly from the series expression for $J_n^{u,v}$.

9. **Applications to Rice's $H_n(\zeta, p, v)$.** S. O. Rice[7] has studied the polynomials

$$H_n(\zeta, p, v) = F(-\,n,\, n + 1,\, \zeta;\, 1,\, p;\, v).$$

Here a ${}_3F_2$ is involved so that eight contiguous function relations appear. Since in the ${}_3F_2$ one parameter is fixed and two others are related,

[6] Loc. cit. p. 575.

[7] Loc. cit. p. 108.

some of the relations are not pertinent; that is, they relate H_n to other functions. All that comes out of the eight relations is the set of four:

$$(24)\quad \begin{aligned}&(\zeta - p + 1)H_n(\zeta, p, v)\\ &\qquad = \zeta H_n(\zeta + 1, p, v) - (p - 1)H_n(\zeta, p - 1, v),\end{aligned}$$

$$(25)\quad \begin{aligned}&(\zeta + n)H_n(\zeta, p, v) + (\zeta - n)H_{n-1}(\zeta, p, v)\\ &\qquad = \zeta H_n(\zeta + 1, p, v) + \zeta H_{n-1}(\zeta + 1, p, v),\end{aligned}$$

$$(26)\quad \begin{aligned}&p[2\zeta - 1 - (\zeta + p - 1)v]H_n(\zeta, p, v)\\ &\qquad = p\zeta(1 - v)H_n(\zeta + 1, p, v) + p(\zeta - 1)H_n(\zeta - 1, p, v)\\ &\qquad\qquad + (n + p)(n - p + 1)vH_n(\zeta, p + 1, v),\end{aligned}$$

$$(26a)\quad \begin{aligned}&pH_n(\zeta, p, v) + pH_{n-1}(\zeta, p, v)\\ &\qquad = (p + n)H_n(\zeta, p + 1, v) + (p - n)H_{n-1}(\zeta, p + 1, v).\end{aligned}$$

The polynomials H_n satisfy the pure recurrence relation

$$(27)\quad \begin{aligned}&n(2n - 3)(p + n - 1)H_n\\ &\qquad = (2n - 1)[(n - 2)(p - n + 1) + 2(n - 1)(2n - 3)\\ &\qquad\quad - 2(2n - 3)(\zeta + n - 1)v]H_{n-1}\\ &\qquad\quad - (2n - 3)[2(n - 1)^2 - n(p - n + 1)\\ &\qquad\quad + 2(2n - 1)(\zeta - n + 1)v]H_{n-2}\\ &\qquad\quad - (n - 2)(2n - 1)(p - n + 1)H_{n-3},\end{aligned}$$

as can be verified directly.

10. **A set of relations alternative to those in §6.** There are times, as in treating Rice's H_n, when it is desirable to have contiguous function relations which omit some one of the $F(\beta_j+)$. Such a set is obtained by making the following replacements, in which

$$V_{j,k} = (\beta_q - \beta_j)W_{j,k}.$$

If $p<q$, replace (20) by

$$(28)\quad (2\alpha_k - \beta_q)F = \alpha_k F(\alpha_k +) + (\alpha_k - \beta_q)F(\alpha_k -) - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.$$

If $p=q$, replace (20) by

$$(29)\quad \begin{aligned}&(2\alpha_k - \beta_q + x)F = \alpha_k F(\alpha_k +) + (\alpha_k - \beta_q)F(\alpha_k -)\\ &\qquad - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.\end{aligned}$$

If $p=q+1$, replace (21) by

$$
\begin{aligned}
&[(2\alpha_k - \beta_q)(1 - x) + (A - B)x]F \\
(30) \qquad &\qquad = \alpha_k(1 - x)F(\alpha_k +) + (\alpha_k - \beta_q)F(\alpha_k -) \\
&\qquad\qquad - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.
\end{aligned}
$$

UNIVERSITY OF MICHIGAN

ON THE GROWTH OF THE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. ROSENBLATT

In a recent paper,[1] N. Levinson gave four theorems concerning the behaviour of the solutions of the differential equation of elastic vibrations

$$d^2x/dt^2 + \phi(t)x = 0 \tag{1}$$

as $t \to +\infty$. It is the purpose of this note to give generalizations of the Theorems I and III of Levinson by making use of certain inequalities concerning homogeneous equations of the first order

$$\frac{dx_i}{dt} + \sum_{k=1}^{n} a_{ik}x_k = 0, \qquad i = 1, \cdots, n. \tag{2}$$

Theorems I and III of Levinson run as follows:

THEOREM I. *If $\alpha(t)$ denotes the integral*

$$\alpha(t) = \int_0^t |\phi(t) - c^2| \, dt, \tag{3}$$

then

$$x(t) = O\{\exp(\alpha(t)/2c)\}. \tag{4}$$

THEOREM III. *If $\alpha(t)$ is $O(t)$ then*

$$\limsup_{t\to\infty} |x(t)\exp(\alpha(t)/2c)| > 0. \tag{5}$$

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[1] *The growth of the solutions of a differential equation*, Duke Math. J. vol. 8 (1941) pp. 1–11.