# THE EXISTENCE OF ANORMAL CHAINS

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**1. Introduction.** Let $\overline{B}$ be a Borel field of subsets of a space $X$, and let $P(x, E)$ be for fixed $x$ a probability measure on $\overline{B}$ and for fixed $E$ a $\overline{B}$-measurable function of $x$. $P(x, E)$ may be considered as representing the transition probability of going from $x$ into $E$ in a single trial. Denote by $\Omega$ the space of sequences $\omega: (x_0, x_1, \cdots)$ where $x_i \in X$ and by $\overline{E}$ the Borel field of subsets of $\Omega$ determined by all sets

$$\{x_i \in E\}, \quad \text{where} \quad E \in \overline{B}, \qquad i = 1, 2, \cdots .$$

Doob [2, pp. 102–103][1] has shown that there exists for each $x \in X$ a probability measure $P_x(S)$ defined on $\overline{E}$ such that for every $P_x$-integrable function $f(x_1, \cdots, x_n)$

$$(1) \int f(\omega) dP_x = \int\int \cdots \int f(x_1, \cdots, x_n) dP(x_{n-1}, x_n) \cdots dP(x, x_1),$$

that $\Omega$ with the measure $P_x$ is a Markoff process, that is, $E(x_1, \cdots, x_n; g) = E(x_n; g)$ where $g = g(x_{n+1}, x_{n+2}, \cdots)$ and the $E$'s denote conditional expectations with respect to the indicated variables, and that $E(x_1, \cdots, x_r; f)$ is the function obtained by carrying out the first $n-r$ integrations in (1).

Write $Q(x, E) = P_x(\limsup \{x_i \in E\})$, so that $Q(x, E)$ represents the probability of entering $E$ infinitely often, starting from $x$. Following Doblin [1, p. 68 et seq.] we make the following definitions for sets of $\overline{B}$: $E$ is *inessential* if $Q(x, E) = 0$ for all $x$, and *essential* otherwise. An essential set is *improperly essential* if it is a denumerable sum of inessential sets, and *absolutely essential* otherwise. A finite or denumerable sum of improperly essential sets is consequently improperly essential. $E$ is *closed* if $P(x, E) = 1$ for all $x \in E$, and a closed set is indecomposable if it does not contain two disjunct non-empty closed subsets. An absolutely essential indecomposable set is said to be *normal* if it contains a closed set which contains no improperly essential subsets and *anormal* otherwise. If $X$ is a normal set, we shall say that the Markoff chain determined by $P(x, E)$ is a normal chain.

Doblin [1] has obtained for normal chains many elegant results which are considerably more complicated for the anormal case. For example [1, p. 81] in the normal case there exists a closed set $G$ such

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[1] Numbers in brackets refer to the references cited at the end of the paper.

that $Q(x, E)=1$ for every essential subset $E$ of $G$ and every $x \in G$; in the anormal case $G$ can no longer be chosen independently of $E$. It is consequently of interest to investigate conditions for normality. Doblin [1, p. 82] has given a quite general sufficient condition for the occurrence of the normal case, and no example of an anormal chain has hitherto been given. The purpose of this paper is to give a simple necessary and sufficient condition for the occurrence of the normal case, consisting merely in the measurability of the function $f(x)$ which represents the probability that, starting from $x$, the point remains indefinitely in an improperly essential set. An example of an anormal chain is also given.

## 2. The normal case.

We restate a result of Doblin [1, p. 80] in the following theorem.

Theorem 1. *If $X$ is indecomposable and absolutely essential, the closed set $\{Q(x, E)=1\}$ is non-empty if and only if $E$ is absolutely essential.*

The following lemma asserts that the probability of entering $E$ infinitely often, starting from $y$ at the $j$th trial, is independent of $j$:

Lemma. *With respect to any $P_x$ measure,*

$$P_x(x_j = y; \limsup \{x_i \in E\}) = Q(y, E). \tag{2}$$

Proof. If $f_{m,n}$ is the characteristic function of the set $E(m, n) = \sum_{m \leqq i \leqq n}\{x_i \in E\}$, it follows from (1) that

$$P_y(E(m, n)) = \int\int \cdots \int f_{m,n} dP(x_{n-1}, x_n) \cdots dP(y, x_1)$$
$$= P_x(x_j = y; E(m+j, n+j)).$$

Letting first $n$, then $m$, become infinite, we obtain (2).

The following theorem, which is new and of some independent interest, asserts that for any two sets $T$ and $E$, unless there are points of $T$ which are practically certain to enter $E$ infinitely often, it is impossible for a point to enter both $T$ and $E$ infinitely often.

Theorem 2. *If $Q(x, E) \leqq a < 1$ for all $x \in T$, then for all $x$*

$$h(x) = P_x(\limsup \{x_i \in E\} \cdot \limsup \{x_i \in T\}) = 0. \tag{3}$$

Proof. Define $E(N, n) = \{x_i \notin E \text{ for } N \leqq i < n, x_n \in E\}$, $T(N, n) = \{x_i \notin T \text{ for } N \leqq i < n, x_n \in T\}$. Now for fixed $N$,

$$h(x) \leqq \sum_{N<n<r} P_x(E(N, n)T(n, r) \limsup \{x_i \in E\}). \tag{4}$$

But

$$P_x(E(N, n)T(n, r) \limsup \{x_i \in E\})$$

$$= \int_{E(N,n)T(n,r)} P_x(x_1, \cdots, x_r; \limsup \{x_i \in E\} dP_x$$

$$= \int_{E(N,n)T(n,r)} Q(x_r, E)dP_x \leqq aP_x(E(N, n)T(n, r),$$

by the lemma and the fact that $x_r$ belongs to $T$ in the domain of integration. Using the last inequality in (4), we obtain

$$h(x) \leqq a \sum_{N<n<r} P_x(E(N, n)T(n, r)) = aP_x\Big( \sum_{N<n<r} E(N, n)T(n, r)\Big).$$

Letting $N$ become infinite we obtain $h(x) \leqq ah(x)$, which implies (3).

Our necessary and sufficient condition for the occurrence of the normal case, given in Theorem 3, is an easy consequence of Theorems 1 and 2.

THEOREM 3. *Let $X$ be absolutely essential and indecomposable. Then $X$ is normal if and only if $f(x) = \text{l.u.b.}_{\text{imp.ess.}E}\, Q(x, E)$ is $\overline{B}$-measurable.*

PROOF. By Theorem 1, for every improperly essential $E$ and every $x$ we have $Q(x, E) < 1$. It follows that $f(x) < 1$ for all $x$, since a denumerable sum of improperly essential sets is improperly essential. If $f(x)$ is measurable, there exists an $a < 1$ such that $T = \{f(x) < a\}$ is absolutely essential; for $X = \sum_{n=1}^{\infty} \{f(x) < 1 - 1/n\}$, and not all these sets can be improperly essential. Denoting by $S$ the closed set $\{Q(x, T) = 1\}$, it follows from Theorem 1 that $S$ is non-empty and from Theorem 2 that $Q(x, E) = 0$ for all $x \in S$ and all improperly essential $E$. In particular if $E \subset S$ and $E$ is not absolutely essential, $Q(x, E) = 0$ for all $x \in E$, which by Theorem 2 in the special case $T = E$ implies that $E$ is inessential. Thus $S$ is a non-empty closed set containing no improperly essential subsets, and $X$ is normal.

Conversely if there exists such a subset $S$ it is easily verified that $f(x) = Q(x, X - S)$ and is therefore measurable.

3. **An anormal chain.** The space $X$ is the semi-infinite interval $0 \leqq x < \infty$, and $\overline{B}$ is the Borel field of all finite or denumerable subsets of $X$ and their complements. Let $a_n$ be any sequence of numbers such that $0 < a_n < 1$, $\prod_0^\infty a_n > 0$. We define

$$P(x, E) = a_n f(x + 1, E) + (1 - a_n)d(E),$$

where $n$ is the largest positive integer not exceeding $x$, $d(E)$ is 1 if $E$ is non-denumerable and 0 otherwise, and $f(x, E)$ is the characteristic function of $E$. $P(x, E)$ is clearly a probability measure on $\overline{B}$ for fixed $x$. To verify that $P(x, E)$ is $\overline{B}$-measurable for fixed $E$, we may assume that $E$ is at most denumerable, since $P(x, CE)=1-P(x, E)$. For this case $P(x, E)=0$ except on an at most denumerable set and is consequently measurable. Since the probability of going from $x$ into $x+1$ is $a_n$ for $n \leqq x < n+1$ and since $\prod_0^\infty a_n > 0$, every set containing all points $x+n$, $n=1, 2, \cdots$, for some $x$ is essential. The closed sets are those nondenumerable sets which contain with $x$ all points $x+n$, so that $X$ is indecomposable. Finally $X$ is absolutely essential, since $\sum S_n = X$ implies that some $S_n$ is nondenumerable and hence esssential. Thus $X$ is anormal.

## References

**1.** Doblin, W. *Chaines simples constantes de Markoff*, Ann. École Norm. vol. 3 (1940).

**2.** Doob, J. L., *Stochastic processes with an integral-valued parameter*, Trans. Amer. Math. Soc. vol. 44 (1938).

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