NOTE ON THE EXPANSION OF A POWER SERIES INTO A CONTINUED FRACTION

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1. **Introduction.** In view of the fact that the continued fraction frequently furnishes a method for summing a slowly convergent or even divergent power series, it is desirable to have a simple algorithem for obtaining the continued fraction. We present here such an algorithm based upon the fact that the process for constructing a sequence of orthogonal polynomials can be so arranged that it gives simultaneously a continued fraction expansion for a power series. It has been known at least since Tschebycheff that the problem of constructing a sequence of orthogonal polynomials is related to the problem of expanding a power series into a continued fraction. However, the fact that the two problems are actually identical does not seem to have been emphasized.

2. **The expansion of a power series into a J-fraction.** A continued fraction of the form

$$\frac{a_0}{b_1+z}-\frac{a_1}{b_2+z}-\frac{a_2}{b_3+z}-\cdots \tag{2.1}$$

is called a *J-fraction*. The a_p and b_p are constants, and z is a complex variable. We shall suppose that the a_p are different from zero. We denote by $A_p(z)$ and $B_p(z)$ the pth numerator and denominator, respectively, of the J-fraction, so that $A_p(z)/B_p(z)$ is its pth approximant. The usual recurrence formulas

$$\begin{aligned} &A_0 = 0,\ A_1 = a_0, \qquad A_p = (b_p + z)A_{p-1} - a_{p-1}A_{p-2}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad p = 2, 3, 4, \cdots, \\ &B_0 = 1,\ B_1 = b_1 + z,\ B_p = (b_p + z)B_{p-1} - a_{p-1}B_{p-2}, \end{aligned} \tag{2.2}$$

show that $A_p(z)$ is a polynomial of degree $p-1$, and $B_p(z)$ is a polynomial of degree p:

$$\begin{aligned} A_p(z) &= \alpha_{p,0}z^{p-1} + \alpha_{p,1}z^{p-2} + \cdots + \alpha_{p,p-1}, \\ B_p(z) &= \beta_{p,0}z^p + \beta_{p,1}z^{p-1} + \cdots + \beta_{p,p}. \end{aligned} \tag{2.3}$$

We note that

Presented to the Society, August 14, 1944; received by the editors February 21, 1944, and, in revised form, June 5, 1944.

$$\beta_{p,0} = 1, \qquad \beta_{p,1} = b_1 + b_2 + \cdots + b_p. \tag{2.4}$$

By means of (2.2) we readily obtain the determinant formula

$$A_p(z)B_{p-1}(z) - A_{p-1}(z)B_p(z) = a_0a_1 \cdots a_{p-1}, \qquad p = 1, 2, 3, \cdots. \tag{2.5}$$

Consequently we find, with the aid of (2.4), that

$$\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = \frac{a_0a_1 \cdots a_n}{z^{2n+1}} + \frac{h_n}{z^{2n+2}} + \cdots$$

where

$$h_n = - a_0a_1 \cdots a_n(b_1 + b_2 + \cdots + b_{n+1}). \tag{2.6}$$

It follows that there exists a power series

$$P(1/z) = c_0/z + c_1/z^2 + c_2/z^3 + \cdots \tag{2.7}$$

such that the expansion in descending powers of z of $A_n(z)/B_n(z)$ agrees term by term with $P(1/z)$ for the first $2n$ terms ($n=1, 2, 3, \cdots$). This uniquely determined power series is called the equivalent power series of the J-fraction.

We shall now write down formulas connecting the various constants, $\alpha_{p,q}$, $\beta_{p,q}$, c_p, a_p and b_p. These formulas serve as an algorithm for expanding a given power series $P(1/z)$ into a J-fraction, and, conversely, for obtaining the equivalent power series of a given J-fraction.

$$\begin{aligned}
&\beta_{00} = 1, \qquad c_0\beta_{00} = a_0, \qquad c_1\beta_{00} = h_0 = - a_0b_1; \\
&b_1 = - h_0/a_0, \qquad (\beta_{10}, \beta_{11}) = (1, b_1), \\
&\qquad (c_2, c_1)\begin{pmatrix}\beta_{10}\\ \beta_{11}\end{pmatrix} = a_0a_1, \qquad (c_3, c_2)\begin{pmatrix}\beta_{10}\\ \beta_{11}\end{pmatrix} = h_1 = - a_0a_1(b_1 + b_2); \\
&b_2 = h_0/a_0 - h_1/a_0a_1, \\
&\qquad (\beta_{20}, \beta_{21}, \beta_{22}) = (\beta_{10}, \beta_{11})\begin{pmatrix}1, & b_2, & 0\\ 0, & 1, & b_2\end{pmatrix} - a_1(0, 0, \beta_{00}), \\
&\qquad (c_4, c_3, c_2)\begin{pmatrix}\beta_{20}\\ \beta_{21}\\ \beta_{22}\end{pmatrix} = a_0a_1a_2, \\
&\qquad (c_5, c_4, c_3)\begin{pmatrix}\beta_{20}\\ \beta_{21}\\ \beta_{22}\end{pmatrix} = h_2 = - a_0a_1a_2(b_1 + b_2 + b_3);
\end{aligned} \tag{2.8}$$

$$b_3 = h_1/a_0a_1 - h_2/a_0a_1a_2,$$

$$(\beta_{30}, \beta_{31}, \beta_{32}, \beta_{33}) = (\beta_{20}, \beta_{21}, \beta_{22}) \begin{pmatrix} 1, & b_3, & 0, & 0 \\ 0, & 1, & b_3, & 0 \\ 0, & 0, & 1, & b_3 \end{pmatrix} - a_2(0, 0, \beta_{10}, \beta_{11}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;$$

$$(2.9) \quad (\alpha_{n,0}, \alpha_{n,1}, \cdots, \alpha_{n,n-1}) = (\beta_{n,0}, \beta_{n,1}, \cdots, \beta_{n,n-1}) \cdot \begin{Bmatrix} c_0, & c_1, & \cdots, & c_{n-1} \\ 0, & c_0, & \cdots, & c_{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & \cdots, & c_0 \end{Bmatrix}, \qquad n = 1, 2, 3, \cdots, (\beta_{n,0} = 1).$$

By way of illustration, we shall obtain the third approximant of the J-fraction for the function $P(1/z) = \log(1+1/z)$. Here $c_p = (-1)^p/(p+1)$, $p = 0, 1, 2, \cdots$. We then have:

$$\beta_{00} = 1, \qquad c_0 = a_0 = 1;$$

$$b_1 = 1/2, \quad (\beta_{10}, \beta_{11}) = (1, 1/2),$$

$$(1/3, -1/2)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = 1/12 = a_1,$$

$$(-1/4, 1/3)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = -1/15 = h_1;$$

$$b_2 = 1/2, \quad (\beta_{20}, \beta_{21}, \beta_{22}) = (1, 1/2)\begin{pmatrix} 1, & 1/2, & 0 \\ 0, & 1, & 1/2 \end{pmatrix} - (1/12)(1, 0, 1)$$

$$= (1, 1, 1/6),$$

$$(1/5, -1/4, 1/3)\begin{pmatrix} 1 \\ 1 \\ 1/6 \end{pmatrix} = 1/180 = a_0a_1a_2, \quad a_2 = 1/15,$$

$$(-1/6, 1/5, 1/4)\begin{pmatrix} 1 \\ 1 \\ 1/6 \end{pmatrix} = -1/120 = h_2;$$

$$b_3 = 1/2, \qquad (\beta_{30}, \beta_{31}, \beta_{32}, \beta_{33}) = (1, 3/2, 3/5, 1/20);$$

$$(\alpha_{30}, \alpha_{31}, \alpha_{32}) = (1, 1, 11/60).$$

Consequently, the third approximant of the J-fraction is

$$\frac{A_3(z)}{B_3(z)} = \frac{1}{1/2+z} - \frac{1/12}{1/2+z} - \frac{1/15}{1/2+z}$$
$$= \frac{z^2+z+(11/60)}{z^3+(3/2)z^2+(3/5)z+(1/20)}.$$

We remark that for $z=1$ this gives $\log 2 = .69312 \cdots$, which is exact to *four* decimal places. Only six coefficients of the power series were used in the computation.

By the same method we find that the seventh approximant of the J-fraction expansion of the divergent power series

$$\frac{B_1}{1\cdot 2\cdot z} - \frac{B_3}{3\cdot 4\cdot z^3} + \frac{B_5}{5\cdot 6\cdot z^5} - \cdots,$$

where $B_1=1/6$, $B_3=1/30$, $B_5=1/42$, $\cdots$ are the Bernoulli numbers, is

$$\frac{1/12}{z} + \frac{1/30}{z} + \frac{53/210}{z} + \frac{195/371}{z} + \frac{22999/22737}{z}$$
$$+ \frac{29944523/19733142}{z} + \frac{109535241009/48264275462}{z}.$$

Stieltjes [**3**, p. 521][1] proved that this J-fraction converges for $R(z)>0$ to the remainder $J(z)$ in Stirling's formula $\log \Gamma(z) = (z-1/2)\log z - z + (1/2)\log(2\pi) + J(z)$. He remarked that the law of formation of the coefficients in the J-fraction seems to be extremely complicated.

3. **Proof of the formulas (2.8) and (2.9).** We shall first prove that the formulas (2.8) constitute an arrangement of the algorithm for constructing a sequence of polynomials $B_n(z) = z^n + \beta_{n,1}z^{n-1} + \cdots + \beta_{n,n}$ which are orthogonal relative to a certain operator S. We define S to be the operator which replaces every z^p by c_p in any polynomial upon which it operates:

$$S(\beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_n) = S(\beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_n z^0)$$
$$= \beta_0 c_n + \beta_1 c_{n-1} + \cdots + \beta_n c_0,$$

where $c_0, c_1, \cdots, c_p, \cdots$ are given constants. Two polynomials B_p and B_q are said to be *orthogonal* if $S(B_pB_q)=0$ when the degrees p and q are unequal. We shall prove the following theorem:

[1] Numbers in brackets refer to the bibliography at the end of the paper.

THEOREM A. *Let m be a positive integer, and put*

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix}, \qquad p = 0, 1, 2, \cdots.$$

There exists a sequence of polynomials $B_n(z) = z^n + \beta_{n,1}z^{n-1} + \cdots + \beta_{n,n}$, $n = 0, 1, 2, \cdots, m$, such that

$$(3.1) \qquad S(B_pB_q)\begin{cases} = 0 & \text{if} \quad p \neq q,\ p \leqq m,\ q \leqq m, \\ \neq 0 & \text{if} \quad p = q < m, \end{cases}$$

if and only if $\Delta_p \neq 0$ for $p = 0, 1, 2, \cdots, m-1$. The polynomials are uniquely determined by the formulas

$$(3.2) \qquad B_{-1} = 0, \quad B_0 = 1, \quad B_p = (b_p + z)B_{p-1} - a_{p-1}B_{p-2}, \qquad p = 1, 2, 3, \cdots, m,$$

where

$$S(z^pB_p) = a_0a_1 \cdots a_p \neq 0,$$
$$(3.3) \qquad S(z^{p+1}B_p) = -a_0a_1 \cdots a_p(b_1 + b_2 + \cdots + b_{p+1}), \qquad p = 0, 1, 2, \cdots, m-1.$$

PROOF. We suppose first that $\Delta_p \neq 0$ for $p = 0, 1, \cdots, m-1$, and shall prove that the required polynomials exist uniquely, and are given recurrently by (3.2) and (3.3). Since $B_0 = 1$, we have: $S(B_0^2) = S(1) = S(z^0) = c_0 = \Delta_0 \neq 0$. Let $B_1 = b_1 + z$. Then, $S(B_1) = b_1c_0 + c_1 = 0$ if and only if

$$S(B_0) = a_0, \qquad S(zB_0) = -a_0b_1.$$

Using induction, suppose that $B_0, B_1, \cdots, B_n$, $n < m$, have been uniquely determined such that (3.1) holds for $p \leqq n$, $q \leqq n$, (3.2) holds for $p \leqq n$, and (3.3) holds for $p \leqq n-1$. Now, an arbitrary polynomial of degree $n+1$ in which the coefficient of z^{n+1} is unity can be expressed uniquely in the form $B_{n+1} = (z + b_{n+1})B_n - a_nB_{n-1} + k_0B_0 + k_1B_1 + \cdots + k_{n-2}B_{n-2}$, where $b_{n+1}, a_n, k_0, k_1, \cdots, k_{n-2}$ are suitable constants. The conditions $S(z^pB_{n+1}) = 0$, $p = 0, 1, \cdots, n-2$, give in succession: $k_0a_0 = 0$, $k_1a_0a_1 = 0, \cdots, k_{n-2}a_0a_1 \cdots a_{n-2} = 0$, so that, since $a_p \neq 0$ for $p = 0, 1, \cdots, n-2$, we must have $k_0 = k_1 = \cdots = k_{n-2} = 0$. From the conditions $S(z^{n-1}B_{n+1}) = 0$ and $S(z^nB_{n+1}) = 0$, we then find that $S(z^nB_n) = a_0a_1 \cdots a_n$ and $S(z^{n+1}B_n) = -a_0a_1 \cdots a_n(b_1 + b_2 + \cdots + b_{n+1})$. Then, from the system of equations: $S(z^pB_n) = 0$, $p = 0, 1, \cdots, n-1$, $S(z^nB_n) = a_0a_1 \cdots a_n$, we find at once that

$$\Delta_n = a_0 a_1 \cdots a_n \Delta_{n-1}, \tag{3.4}$$

and, inasmuch as $n<m$, we see that $a_n \neq 0$. Consequently, B_{n+1} is uniquely determined, and (3.2), (3.3) hold for $p=n+1$ and $p=n$, respectively. Also, $S(B_pB_q)=0$ for $p\neq q$, $p\leqq n+1$, $q\leqq n+1$. Moreover, if $n+1<m$, then $S(B_{n+1}^2)=S(z^{n+1}B_{n+1})\neq 0$, for otherwise we would have $\Delta_{n+1}=0$. We have proved that the condition $\Delta_p\neq 0$, $p=0, 1, \cdots, m-1$, is *sufficient* for the polynomials to exist (uniquely) and satisfy the stated conditions.

Conversely, the condition is *necessary*. For, it is obviously necessary that $\Delta_0=c_0\neq 0$; and if $S(z^pB_n)=0$, for $p=0, 1, 2, \cdots, n-1$, $S(z^nB_n)=g_n\neq 0$, $n<m$, then the relation $\Delta_n=g_n\Delta_{n-1}$ must hold, and hence $\Delta_p\neq 0$, $p=0, 1, 2, \cdots, m-1$.

One will now readily see that the polynomials B_p given by (3.2) and (3.3) are the same as those given by (2.8).

THEOREM B. *Let $\Delta_p\neq 0$, $p=0, 1, 2, \cdots$, and define polynomials $A_n(z)=\alpha_{n,0}z^{n-1}+\alpha_{n,1}z^{n-2}+\cdots+\alpha_{n,n-1}$ by means of* (2.9). *Then,*

$$\frac{A_n(z)}{B_n(z)} = \frac{a_0}{b_1+z} - \frac{a_1}{b_2+z} - \cdots - \frac{a_{n-1}}{b_n+z}, \quad n = 1, 2, 3, \cdots, \tag{3.5}$$

and we have the formal power series identity

$$P(1/z)B_n(z) - A_n(z) = \frac{a_0a_1\cdots a_n}{z^{n+1}} + \frac{h_n}{z^{n+2}} + \cdots, \tag{3.6}$$

where $h_n=-a_0a_1\cdots a_n(b_1+b_2+\cdots+b_{n+1})$ and $P(1/z)=\sum(c_p/z^{p+1})$.

PROOF. Let us *define* polynomials $A_n(z)$ by means of the formulas $A_{-1}=-1$, $A_0=0$, $A_p=(b_p+z)A_{p-1}-a_{p-1}A_{p-2}$, $p=1, 2, 3, \cdots$. From these recurrence formulas and (3.2) it follows that (3.5) holds. Furthermore, we may conclude from the determinant formula (2.5) that there exists a power series $P^*(1/z)=\sum(c_p^*/z^{p+1})$ such that

$$P^*(1/z)B_n(z) - A_n(z) = \frac{a_0a_1\cdots a_n}{z^{n+1}} + \frac{h_n}{z^{n+2}} + \cdots.$$

On equating coefficients of corresponding powers of z on either side of this identity we find that precisely the relations (3.3) hold but with c_p replaced by c_p^*. Inasmuch as those relations determine the c_p uniquely in terms of the a_p and b_p, we conclude that $c_p^*=c_p$, $p=0, 1, 2, \cdots$, or $P^*(1/z)=P(1/z)$, so that (3.6) holds. The relation (2.9) may now be obtained by equating the coefficients of $z^0, z^1, \cdots, z^{n-1}$ on either side of the identity (3.6).

This completes the proof of the formulas (2.8) and (2.9) connecting the constants $\alpha_{p,q}$, $\beta_{p,q}$, c_p, a_p, b_p of a J-fraction and its equivalent power series.

4. **The expansion of a power series into an S-fraction.** If we replace z by $1/z$ in the power series (2.7) and in its J-fraction expansion (2.1), the series becomes

$$P(z) = c_0 z + c_1 z^2 + c_2 z^3 + \cdots, \tag{4.1}$$

and the J-fraction becomes

$$\frac{a_0 z}{1 + b_1 z} - \frac{a_1 z^2}{1 + b_2 z} - \frac{a_2 z^2}{1 + b_3 z} - \cdots. \tag{4.2}$$

An important special case arises when all the b_p are equal to zero. For, in this case it is evident that $P(z)/z$ contains only *even* powers of z. If we change the notation and replace c_{2n} by c_n, we see that the power series

$$c_0 z + 0z^2 + c_1 z^3 + 0z^4 + c_2 z^5 + \cdots \tag{4.3}$$

has the expansion

$$\frac{a_0 z}{1} - \frac{a_1 z^2}{1} - \frac{a_2 z^2}{1} - \cdots. \tag{4.4}$$

Let us now remove a factor z from both (4.3) and (4.4), and subsequently replace z^2 by z. Afterwards, we again multiply both the series and continued fraction by z and then replace z by $1/z$. The series then becomes

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots, \tag{4.5}$$

and the continued fraction becomes

$$\frac{a_0}{z} - \frac{a_1}{1} - \frac{a_2}{z} - \frac{a_3}{1} - \frac{a_4}{z} - \cdots. \tag{4.6}$$

Conversely, if the power series (4.5) has a continued fraction expansion of the form (4.6), then the power series (4.3) has a continued fraction expansion of the form (4.2) in which the b_p are all equal to zero. We shall call (4.6) an *S-fraction* since it is the form of continued fraction preferred by Stieltjes.

From the preceding it follows that the condition for (4.5) to have an S-fraction expansion (4.6) in which $a_p \neq 0$, $p = 0, 1, 2, \cdots$, is the same as the condition for (4.3) to have an expansion (4.2). This condition is that the determinants

$$c_0, \quad \begin{vmatrix} c_0, & 0 \\ 0, & c_1 \end{vmatrix}, \quad \begin{vmatrix} c_0, & 0, & c_1 \\ 0, & c_1, & 0 \\ c_1, & 0, & c_2 \end{vmatrix}, \quad \begin{vmatrix} c_0, & 0, & c_1, & 0 \\ 0, & c_1, & 0, & c_2 \\ c_1, & 0, & c_2, & 0 \\ 0, & c_1, & 0, & c_2 \end{vmatrix}, \cdots$$

be different from zero. From this we readily conclude the well known result that the power series (4.5) has an S-fraction expansion if and only if the determinants

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix}, \qquad \Omega_p = \begin{vmatrix} c_1, & c_2, & \cdots, & c_{p+1} \\ c_2, & c_3, & \cdots, & c_{p+2} \\ \cdot & \cdot & \cdot & \cdot \\ c_{p+1}, & c_{p+2}, & \cdots, & c_{2p+1} \end{vmatrix}$$

$$(p = 0, 1, 2, \cdots)$$

are all different from zero.

It is immediately evident that the algorithm of §2 can be used to compute the coefficients in (4.6) if we there replace c_{2n} by c_n and c_{2n+1} by 0.

5. **A theorem of Stieltjes.** A remarkable formulation of the problem of expanding a power series into a continued fraction was given by Stieltjes [3, p. 184]. Rogers [2] rediscovered part of the result of Stieltjes in a slightly different form. We offer the following formulation of the theorem.

The problem of expanding the power series

$$\frac{1}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots$$

into a continued fraction

$$\frac{1}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_3 + z} - \cdots$$

is equivalent to the problem of securing a power series identity of the form

$$Q(x + y) = Q(x)Q(y) + a_1Q_1(x)Q_1(y) + a_1a_2Q_2(x)Q_2(y) + \cdots,$$

where $a_p \neq 0$, $p=1, 2, 3, \cdots$,

$$Q_n(z) = \frac{z^n}{n!} + \pi_{n,n+1}\frac{z^{n+1}}{(n+1)!} + \pi_{n,n+2}\frac{z^{n+2}}{(n+2)!} + \cdots,$$

and

$$Q_0(z) = Q(z) = 1 + c_1\frac{z}{1!} + c_2\frac{z^2}{2!} + c_3\frac{z^3}{3!} + \cdots.$$

The formulas connecting the various constants are:

$$\pi_{0,0} = 1, \qquad \pi_{p,q} = 0 \quad \text{for} \quad p > q;$$

$$(\pi_{0,q}, \pi_{1,q}, \pi_{2,q}, \cdots)$$
$$= (\pi_{0,q-1}, \pi_{1,q-1}, \pi_{2,q-1}, \cdots)\begin{Vmatrix} b_1, & 1, & 0, & 0, \cdots \\ a_1, & b_2, & 1, & 0, \cdots \\ 0, & a_2, & b_3, & 1, \cdots \\ \cdot & \cdot & \cdot & \cdot \cdots \end{Vmatrix};$$

$$b_1 = \pi_{0,1}, \qquad b_p = \pi_{p-1,p} - \pi_{p-2,p-1}, \qquad p = 2, 3, 4, \cdots;$$

$$c_{p+q} = \pi_{0,p}\pi_{0,q} + a_1\pi_{1,p}\pi_{1,q} + a_1a_2\pi_{2,p}\pi_{2,q} + \cdots.$$

This combines the idea of Rogers with a formulation of Stieltjes' algorithm particularly adapted to the J-fraction. A part of this is given in [1, pp. 328–329]. We omit the proof.

Both Stieltjes and Rogers gave the example:

$$\int_0^\infty \operatorname{sech}^k u\, e^{-zu}du = \frac{1}{z+}\,\frac{1\cdot k}{z+}\,\frac{2(k+1)}{z+}\,\frac{3(k+2)}{z+\cdots}.$$

This can be obtained almost by inspection from the identity

$$\operatorname{sech}^k (x+y) = (\cosh x \cosh y + \sinh x \sinh y)^{-k}.$$

Bibliography

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