is in H whenever $a, b \in H$, and $x(yH) = (xy)H$ for every $x, y \in Q$. Under these circumstances Q has an expansion in left cosets of H, and the system Q/H of left cosets is made into a left loop under a suitable definition of multiplication. The group $\mathfrak{L}$ spanned by the left multiplications (that is, the permutations $L_x(a) = ax$) of Q is introduced, and an isomorphism of Q with $\mathfrak{L}/\mathfrak{L}_0$ (where $\mathfrak{L}_0$ is the subgroup of $\mathfrak{L}$ consisting of all permutations keeping the identity E fixed) is established. It is shown that the admissible left subloops of Q are in 1:1 correspondence with the subgroups $\mathfrak{M} \supset \mathfrak{L}_0$ of $\mathfrak{L}$, and isomorphisms $H \cong \mathfrak{M}/\mathfrak{L}_0$, $Q/H \cong \mathfrak{L}/\mathfrak{M}$ are established. An extension theory is developed: given left loops H and K, a construction is given for all left loops Q such that $Q/H = K$. Necessary and sufficient conditions are given (when H is a group) that Q shall be a group, and specialization of H to be normal yields the Schreier extension theory. (Received October 20, 1944.)

12. Seymour Sherman: *Complex polynomials and polygonal domains.*

Theorems of Sturm, Routh, and Hurwitz have been generalized so as to provide a finite numerical algorithm for finding the number of such roots of a polynomial with complex coefficients as lie on a generalized polygon or linear transformation thereof. By this means a finite procedure is given for determining the number of roots of a polynomial lying in a quadrant, half-plane, circle, or circular sector. Such problems have proved of interest recently in connection with airplane flutter (S. Sherman, Jane DiPaola, and H. Frissel, *Routh's discriminant, flutter, and ground resonance,* abstract 50-7-190) and econometric business cycle analysis (P. A. Samuelson, *Conditions that the roots of a polynomial be less than unity in absolute value,* Annals of Mathematical Statistics vol. 12 (1941)). (Received October 17, 1944.)

Analysis

13. E. F. Beckenbach: *A Looman-Menchoff theorem for Newtonian vectors.*

It is shown that if the vector function $X(x, y, z)$ is continuous in the finite domain D, if except at most at the points of a denumerable set of points in D, $X(x, y, z)$ is totally differentiable in the planes parallel to the coordinate planes, and if the curl and divergence of $X(x, y, z)$ vanish almost everywhere in D, the $X(x, y, z)$ has continuous partial derivatives of all orders. (Received October 28, 1944.)

14. R. E. Fullerton: *Linear operators with range in a space of differentiable functions.*

The Banach space $C^n(0, 1)$ is defined to be the space of functions possessing n continuous derivations over the interval (0, 1) with norm $\|f\| = \text{l.u.b.}_{0 \leq t \leq 1}\text{l.u.b.}_{k \leq n}|f^{(k)}(t)|$. If $Tx = f$ is a bounded linear operator from a Banach space X to $C^n(0, 1)$, Tx is representable in the form $\bar{x}_t x$ where $\bar{x}_t$ is a function defined from (0, 1) to the space $\bar{x}$ conjugate to X. In this paper, necessary and sufficient conditions that $\bar{x}_t$ represent such an operator are found. Both bounded and completely continuous operators are investigated. Particular attention is devoted to representations of operators from sequence spaces and Lebesgue spaces to the space $C^n(0, 1)$. In all cases the expression for the norm of the operator is obtained in terms of the function $\bar{x}_t$. (Received October 20, 1944.)

15. Einar Hille: *The commutative n-parameter case of semi-groups of linear transformations.*

Let $T(s)$ be a family of linear bounded transformations on a B-space $\mathfrak{X}$ to itself such that $T(s)T(t)=T(s+t)$. Here $s=\{s_1, s_2, \cdots, s_n\}$ and $T(s)$ is defined in S: $0\leqq s_k<\infty$, $\sum s_k>0$; parameter addition being vector addition. If $T(s)$ is strongly measurable in S then it is strongly continuous. If $\|T(s)\|\leqq M$, $\sum s_k<1$, and if $U_s T(s)[\mathfrak{X}]$ is dense in $\mathfrak{X}$, then $T(s)\rightarrow I$ when $s\rightarrow 0$. From this it follows that $T(s)$ is the direct product of not more than n one-parameter commuting semi-groups. Conversely the direct product of such semi-groups is a semi-group of the type considered here. If $A_1, A_2, \cdots, A_n$ are the infinitesimal generators of the factors, then the Lie ring of $T(s)$ is the set $\sum \alpha_k A_k$ with $\alpha_k\geqq 0$. (Received October 28, 1944.)

16. H. K. Hughes: *The asymptotic developments of a class of entire functions.*

In an earlier paper (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 425–430) the author established a general theorem which furnishes the asymptotic expansions, valid for large $|z|$, of an entire function defined by a Maclaurin series of the form $\sum_{n=0}^{\infty} g(n)z^n$, where the coefficient $g(n)$ of z^n satisfies certain conditions. In the present paper, the theorem is applied to find the asymptotic expansions of the function $f(z)$ for which $g(n)$ is the reciprocal of the product of m gamma functions of the form $\Gamma(\alpha_j n+a_j)$; $1\leqq j\leqq m$; $m\geqq 2$; $\alpha_j>0$; a_j any constants, real or complex. The main part of the paper consists of showing that the present $g(n)$ satisfies the condition stated in the hypotheses of the theorem. This requires the introduction of a special lemma concerning the asymptotic representation of a fraction whose numerator and denominator are each the product of m gamma functions of the type described above, with certain relations existing among the constants. (Received October 13, 1944.)

17. C. T. Loo: *Note on the strong summability of Fourier series.*

Let $f(x)$ be of period 2π and of the class L^p, $p>1$. Let $k=2, 3, \cdots$ be an integer less than p and suppose that for some x_0 and for some S, $\int_0^t|f(x_0+u)+f(x_0-u)-2S|^p du=o(t^p)$ as $t\rightarrow 0$. Then the partial sums $S_n(x)$ of the Fourier series of f satisfy the relation $\sum_{m=0}^{n}|S_{m^k}(x_0)-S|^2=o(n)$. (Received October 16, 1944.)

18. C. T. Loo: *Two Tauberian theorems in the theory of Fourier series.*

Let $f(x)$ be integrable L and of period 2π. Let $a_0/2+\sum_1^{\infty}(a_n \cos nx+b_n \sin nx)$ $\equiv\sum_0^{\infty}A_n(x)$ be the Fourier series of f, and let $\sigma_n^{\alpha}(x)$ denote the αth Cesàro means of this series. Let, for some fixed x_0 and S, $\phi(t)=\{f(x_0+t)+f(x_0-t)-2S\}/2$, $\phi_\alpha(t)=\alpha t^{-\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)du$. Then (i) if $\alpha>0$ and if $\sigma_n^{\alpha}(x_0)-S=o(1/\log n)$, we have $\phi_{1+\alpha}(t)\rightarrow 0$; (ii) if $\sigma_n^{\alpha}(x_0)-S=O(n^{-\epsilon})$ and if $A_n(x_0)=O(n^{-\delta})$ with $\alpha>0$, $\epsilon>0$, $\delta>0$, $\epsilon+\delta>1$, we have $\phi_\alpha(t)\rightarrow 0$. (Received October 16, 1944.)

19. W. T. Reid: *A matrix differential equation of Riccati type.*

This paper is concerned with the matrix differential equation (I): $W'+WA(x)+D(x)W+WB(x)W=C(x)$, where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are given $n\times n$ square matrices whose elements are continuous functions on the interval $a\leqq x\leqq b$. The first part of the paper contains generalizations of well known theorems on solutions of a single ordinary differential equation of Riccati type; in particular, the theorems established provide decided extensions of results proved by W. M. Whyburn (Amer. J. Math. vol. 56 (1934) pp. 587–592) for the equation $W'+WW=C(x)$. In the second part of the paper the matrix equation (I) is used to present a compact discussion of the Legendre transformation of the second variation for simple integral problems of the calculus of variations. (Received October 19, 1944.)

20. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* II.

The relations stated in abstract 50-7-185 are generalized as follows: 1. Autodistributivity: $f(u_1, u_2, \cdots, u_{n+1})=f(u, t_1, t_2, \cdots, t_n)$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$ and $u=f(x_1, x_2, \cdots, x_{n+1})$. 2. Restricted distributivity: $f(v_0, v_1, v_2, \cdots, v_n)=\phi(v, t_1, t_2, \cdots, t_n)$ where $v_0=\phi(x, t_1, t_2, \cdots, t_n)$, $v_i=\phi(t_i, t_1, t_2, \cdots, t_n)$ and $v=f(x, t_1, t_2, \cdots, t_n)$. 3. Reflexiveness: $f(x, x, \cdots, x)=x$. 4. Duals, in f and ϕ, of the preceding postulates. 5. Closure of the set S of elements x_i under the compositions f and ϕ. Various systems associated with the above postulates are discussed. For example: 6. The postulate $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\}=f(t_1, t_2, \cdots, t_n)$ and its dual are adjoined, with definition: $f(x_1)=x_1$, $f(x_1, x_2)=f\{x_1, f(x_2)\}, \cdots, f(x_1, x_2, \cdots, x_{n+1})=f\{x_1, f(x_2, \cdots, x_{n+1})\}$, and dually for ϕ. 7. Postulate 2 is replaced by full distributivity. Postulates 1–6 are valid in the domain of the algebra of logic if $f(x_1, x_2)=x_1\cdot x_2$ and $\phi(x_1, x_2)=x_1+x_2$. In particular, any associative, commutative and reflexive function $f(x_1, x_2)$ is autodistributive. (Received October 20, 1944.)

21. W. S. Snyder: *Derivatives of set functions.*

Let Γ be a class of subsets T of the xy-plane, subject to the following conditions: (A) The sets T lie in a bounded portion of the xy-plane. (B) $|T|=|\overline{T}|>0$ for every set T. (C) The parameter of regularity of each T exceeds a fixed positive number ρ. Let f be a real-valued set function defined on Γ. The upper (lower)-derivatives of f at a point p are defined by $u(f, p)=\lim \sup_{|T|\to 0}(f(T)/|T|)$, $p\in T$ $(l(f, p)=-u(-f, p))$. The derivatives are shown to be measurable point functions. Necessary and sufficient conditions of an $\epsilon-\delta_\epsilon$ variety are obtained (a) for the derivatives to be summable, (b) for a unique, finite derivative to exist almost-everywhere, (c) for f to be an indefinite Lebesgue integral. Similar theorems are proved for functions defined on nonregular classes. The results are then extended to several other varieties of derivatives, in particular to the derivative considered by R. C. Young (*Functions of* $\Sigma, \cdots,$ Math. Ann. vol. 29 (1928) pp. 171–216). (Received October 20, 1944.)

22. W. J. Trjitzinsky: *Singular integral equations with Cauchy kernels.*

The theory of equations $\alpha(t)\phi(t)+\int\kappa(t, y)\phi(y)dy/(y-t)=f(t)$ (α, f, κ of Hölder class along L, integrations in sense of Cauchy principal values over L) in recent times has been brought to a high degree of perfection, notably by M. I. Mushelishvili and I. Vecoua, in the case when L consists of a finite number of suitably regular open or closed arcs (in the complex plane of y) without common points. In the present work the theory is developed in the more general situation, presenting substantial new difficulties, when the arcs may have common points. (Received October 20, 1944.)

23. Antoni Zygmund: *On smooth functions.*

Let $f(x)$ be continuous, of period 2π. It is *uniformly smooth* (symbolically: $f\in\lambda^*$), if

$\Delta_h^2 f(x) = f(x+h) + f(x-h) - 2f(x) = o(h)$ uniformly in x. If $\Delta_h^2 f = O(h)$, $f \in \Lambda^*$. If $f \in \text{Lip } 1$, then $f \in \Lambda^*$. The converse is false, since there are $f \in \Lambda^*$ nowhere differentiable. The modulus of continuity of an $f \in \Lambda^*$ is $O(\delta \log \delta)$. The class Λ^* is sometimes more natural than Lip 1. (i) A necessary and sufficient condition that the best approximation $E_n[f] = O(n^{-k})$ $(k = 1, 2, \cdots)$ is $f^{(k-1)} \in \Lambda^*$. (ii) If $f \in \Lambda^*$, so does the conjugate function $\tilde{f}$. (iii) Let f^α, f_α denote the αth derivative and integral of f $(0 < \alpha < 1)$. If $f \in \Lambda^*$, then $f^\alpha \in \text{Lip } (1-\alpha)$. If $f \in \text{Lip } \alpha$ then $f_{1-\alpha} \in \Lambda^*$. (iv) A necessary and sufficient condition that a harmonic function $f(r, x)$, $0 \leq r < 1$, be the Poisson integral of an $f \in \Lambda^*$ is $\partial^2 f(r, x)/\partial x^2 = O\{1/(1-r)\}$. If $\Delta_h^2 f(x) = o(h)$ for each x, f is *smooth* $(f \in \lambda)$. A $g(x)$ defined over a set E is said to satisfy condition D, if $g(x)$ takes all intermediate values. (v) If $f \in \lambda$, $f'(x)$ exists in a dense set (Rajchman) and satisfies condition D. (vi) The sum of a trigonometric series with coefficients $o(1/n)$ satisfies condition D. (vii) If $g(x)$ is continuous, $\tilde{g}(x)$ satisfies condition D. (Received October 28, 1944.)

Applied Mathematics

24. C. H. Dix, C. Y. Fu, Mrs. E. W. McLemore: *The cubic Rayleigh wave equation.*

Consider a plane compressional wave incident on the free plane surface of a semi-infinite elastic medium. The incident and reflected compressional amplitudes are respectively A and B. Then $B/A = N(i, s)/D(i, s)$ where $s = \lambda/\mu$, $ND = 16(s+1)w^3 - 8(3s+4)w^2 + 8(s+2)w - (s+2)$ and $w = \sin^2 r_1$, r_1 = reflection angle of shear wave. Zeros of ND show two or no i's (also zeros of N) corresponding to no reflected compressional wave. The third zero is a zero of D and gives the reciprocal of the important solution of the Rayleigh wave cubic. The cubic curves all pass through one fixed point whose coordinates are $w = 1.0957$, $ND = -1.83927$. For a fixed s, ND has the same value when $w = 0$ that it has when $w = 1$. If s corresponds to a small Poisson ratio then i for the larger zero of N will be very close to 90°, giving no reflection of compressional type whereas for $i = 90°$ all reflected energy is compressional. There is a discontinuity when $s = 0$ where $B/A = +1$ while $B/A = -1$ for s greater than 0. (Received October 27, 1944.)

25. H. W. Eves: *A geometrical note on the isocenter.*

Consider a central projection of plane p on plane p', L being the center of projection, and adopt the convention that angular directions on p (or p') are positive if they are counterclockwise when p (or p') is viewed from L. A point on p is called a positive isocenter on p if all angles on p having the point for vertex are invariant under the projection, and a point on p is called a negative isocenter if all angles on p having the point for vertex project into equal but oppositely directed angles on p'. It is shown that a tilted photograph possesses one and only one positive isocenter and one and only one negative isocenter. These points are geometrically located on the picture. Now the positive isocenter has long been known in photogrammetry, but the existence of the negative isocenter does not seem to have been noticed before. With the combined aid of these two isocenters a simple graphical procedure is developed for rectifying a tilted photograph. The mapping process can then be continued by the method of radial plotting. (Received October 19, 1944.)

26. H. W. Eves: *Analytical and graphical rectification of a tilted photograph.*

An aerial photograph fails to be a perfect map of the ground photographed be-