CONGRUENCES INVOLVING THE PARTITION FUNCTION $p(n)$

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1. Introduction. The purpose of this note is to give certain congruence properties of $p(n)$, the partition function, for moduli 13 and 17, analogous to those obtained by Ramanujan for moduli 5, 7 and 11. The method and notation employed are essentially those of Ramanujan in [1].[1] Let

$$P = P(x) = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n},$$

$$Q = Q(x) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n},$$

$$R = R(x) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n},$$

$$f(x) = \prod_{i=1}^{\infty} (1 - x^i).$$

Then it is known that

$$f(z) = 1 - x - x^2 + x^5 + x^7 - \cdots$$

$$(1) \qquad = \sum_{n=-\infty}^{+\infty} (-1)^n x^{n(3n-1)/2},$$

$$(2) \qquad Q^3 - R^2 = 1728x[f(x)]^{24}.$$

Furthermore, let $\Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n$, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. In particular

$$\Phi_{0,s}(x) = \sum_{n=1}^{\infty} \frac{n^s x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_s(n)x^n,$$

so that

$$P = 1 - 24\Phi_{0,1}(x),$$

$$Q = 1 + 240\Phi_{0,3}(x),$$

$$R = 1 - 504\Phi_{0,5}(x).$$

Then in terms of the functions $\sum_{r,s}(n)$, defined by

$$\sum\nolimits_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m),$$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

the main result for modulus 13 is given by the congruence (10) of §2,

(10) $$p(13n-7)-p(13n-176)-\cdots\equiv-2\Sigma_{5,5}(13n) \pmod{13},$$

and for modulus 17 by the congruence (14) of §3,

(14) $$p(17n-12)-p(17n-301)-\cdots\equiv\Sigma_{5,9}(17n) \pmod{17}.$$

Another interesting congruence for modulus 13 is found in (12) of §2, namely

(12) $$p(13n-7)\equiv-2\sum_{m=1}^{n}\tau(m)p\left(\frac{n-m}{13}\right) \pmod{13},$$

where $\tau(m)$ is the Ramanujan function defined by

(3) $$\sum_{m=1}^{\infty}\tau(m)x^m = x[f(x)]^{24},$$

that is, $\tau(m)$ is the coefficient of x^m in the expansion of $x[f(x)]^{24}$.

2. **Congruence properties, modulus 13.** We let I denote a power series in x with integral coefficients. Now $\Phi_{r,s}(x)=\mathfrak{J}^r\Phi_{0,s-r}(x)$, where $\mathfrak{J}=xd/dx$. Also $\mathfrak{J}P=-24\Phi_{1,2}(x)$, $\mathfrak{J}Q=240\Phi_{1,4}(x)$, and $\mathfrak{J}R=-504\Phi_{1,6}(x)$. Then from Tables I, II and III of [2] we may form the following tables for expressing $\Phi_{r,s}(x)$ in terms of P, Q, and R. Each entry has been reduced modulo 13 so that to each must be added $13I$. Use has also been made of the relations

(4) $$Q^3 = 3R^2-2+13I,$$

(5) $$Q^2R = P+13I.$$

TABLE I

$$\Phi_{0,1}(x) = -6P+6$$
$$\Phi_{0,3}(x) = -2Q+2$$
$$\Phi_{0,5}(x) = -4R+4$$
$$\Phi_{0,7}(x) = -Q^2+1$$
$$\Phi_{0,9}(x) = 3QR-3$$
$$\Phi_{1,2}(x) = 6P^2-6Q$$
$$\Phi_{1,4}(x) = 5R-5PQ$$
$$\Phi_{1,6}(x) = -2PR+2Q^2$$
$$\Phi_{1,8}(x) = -5PQ^2+5QR$$
$$\Phi_{1,10}(x) = -4PQR+R^2+3$$

TABLE II

$$\Phi_{2,3}(x) = P^3 - 3PQ + 2R$$
$$\Phi_{2,5}(x) = -P^2Q + 2PR - Q^2$$
$$\Phi_{2,7}(x) = P^2R - 2PQ^2 + QR$$
$$\Phi_{2,9}(x) = 6P^2Q^2 + PQR + 4R^2 + 2$$
$$\Phi_{2,11}(x) = 5P^2QR + 4P + 4PR^2$$
$$\Phi_{3,4}(x) = -3P^4 + 5P^2Q + 2PR - 4Q^2$$
$$\Phi_{3,6}(x) = 6P^3Q - 5P^2R + 5PQ^2 - 6QR$$
$$\Phi_{3,8}(x) = 5P^3R - 2P^2Q^2 + 2PQR + 3R^2 + 5$$
$$\Phi_{3,10}(x) = 5P^3Q^2 - 2P^2QR - 3PR^2$$
$$\Phi_{3,12}(x) = 5P^3QR - 6P^2 + 6P^2R^2 + 4QR^2 + 4Q$$
$$\Phi_{4,5}(x) = -P^5 - 3P^3Q + 6P^2R + 2PQ^2 - 4QR$$
$$\Phi_{4,7}(x) = -3P^4Q - P^3R - 5P^2Q^2 - PQR - 3$$
$$\Phi_{4,9}(x) = -6P^4R - 2P^3Q^2 + 3P^2QR - 4PR^2 - 4P$$
$$\Phi_{4,11}(x) = -3P^4Q^2 - P^3QR + P^2R^2 + 6P^2 - 3QR^2$$
$$\Phi_{5,6}(x) = 5P^6 + 3P^4Q + 5P^3R - 4P^2Q^2 + 3PQR - 4R^2 + 5$$

Now let $\Delta = (Q^3 - R^2)$. Then by the use of (4) and (5) we may express Δ^7 in terms of P, Q, and R as follows:

$$\Delta^7 = -5P^6 - 2P^4Q + 6P^3R - 6P^2Q^2 - 6PQR - 2R^2 + 2 + 13I.$$

Then, making use of Tables I and II,

$$\Delta^7 = -\Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x) - 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,5}(x) + 2[\Phi_{0,5}(x)]^2 + 13I.$$

But since

$$\Phi_{0,5}(x) = \sum_{n=1}^{\infty} \sigma_5(n) x^n,$$

we have

$$[\Phi_{0,5}(x)]^2 = \left[\sum_{n=1}^{\infty} \sigma_5(n) x^n\right]\left[\sum_{m=1}^{\infty} \sigma_5(m) x^m\right] = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m) x^n,$$

and then

$$(6)\qquad \begin{aligned}\Delta^7 = &-\Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x)\\ &- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,5}(x)\\ &+ 2\sum_{n=2}^{\infty} (\Sigma'_{5,5}(n))x^n + 13I,\end{aligned}$$

where[2]

$$\Sigma'_{5,5}(n) = \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Instead of equation (6) we may obtain a slightly different form for Δ^7 by introducing Ramanujan's function $\tau(n)$, defined by equation (3). Then

$$Q^3 - R^2 = (3R^2 - 2 - R^2) + 13I = 2(R^2 - 1) + 13I$$

and

$$\begin{aligned}6(R^2 - 1) &= -3x[f(x)]^{24} + 13I\\ &= -3\sum_{n=1}^{\infty} \tau(n)x^n + 13I,\end{aligned}$$

so that

$$(7)\qquad \begin{aligned}\Delta^7 = &-\Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x)\\ &- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x)\\ &- 3\sum_{n=1}^{\infty} \tau(n)x^n + 13I.\end{aligned}$$

Now, by (2),

$$\begin{aligned}\Delta^7 = (Q^3 - R^2)^7 &= \{1728x[f(x)]^{24}\}^7\\ &= -x^7f(x^{169})/f(x) + 13I.\end{aligned}$$

But by (1),

$$\begin{aligned}\frac{x^7f(x^{169})}{f(x)} &= x^7\left\{\sum_{n=-\infty}^{+\infty} (-1)^n x^{169n(3n-1)/2}\right\}\left\{\sum_{m=0}^{\infty} p(m)x^m\right\}\\ &= \sum_{n=7}^{\infty} \{p(n-7) - p(n-176) - p(n-345) + \cdots\}x^n,\end{aligned}$$

where the numbers 7, 176, 345, $\cdots$, are alternately of the form $(13m-2)(39m-7)/2$ and $(13m+2)(39m+7)/2$. We therefore have

[2] Ramanujan defines $\Sigma_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m)$ so that $\Sigma_{5,5}(n) = \Sigma'_{5,5}(n) + 2\sigma_5(0)\sigma_5(n)$, $\sigma_5(0) = \zeta(-5)/2 = -1/504$.

$$(8)\qquad \Delta^7 = -\sum_{n=7}^{\infty}\{p(n-7)-p(n-176)-p(n-345)+\cdots\}x^n+13I.$$

Comparing coefficients of x^n in (6) and (8) we obtain

$$(9)\qquad \begin{aligned} p(n-7) - p(n-176) - p(n-345) + \cdots \\ \equiv + n^5\sigma_1(n) - 4n^4\sigma_3(n) - 3n^3\sigma_5(n) \\ + 6n^2\sigma_7(n) - 3n\sigma_9(n) + 3\sigma_5(n) \\ - 2\Sigma'_{5,5}(n) \pmod{13}. \end{aligned}$$

If we replace n by $13n$ we get

$$(10)\qquad \begin{aligned} p(13n-7) - p(13n-176) - \cdots \\ \equiv 3\sigma_5(13n) - 2\Sigma'_{5,5}(13n) \pmod{13} \\ \equiv - 2\Sigma_{5,5}(13n) \pmod{13}. \end{aligned}$$

In a similar fashion, from (7) and (8) we obtain

$$(11)\qquad p(13n-7) - p(13n-176) - \cdots \equiv 3\tau(13n) \pmod{13}.$$

We next wish to compare this result with that obtained by Zuckerman [3]. For this we rewrite (11) as follows

$$\sum_{n=1}^{\infty} p(13n-7)x^{13n}\sum_{n=-\infty}^{+\infty}(-1)^n x^{169n(3n+1)/2} = 3\sum_{n=1}^{\infty}\tau(13n)x^{13n} + 13I.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} p(13n-7)x^{13n} &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\frac{1}{f(x^{169})} + 13I \\ &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\sum_{n=0}^{\infty} p\left(\frac{n}{13}\right)x^{13n} + 13I, \end{aligned}$$

$$p(13n-7) \equiv 3\sum_{m=1}^{n}\tau(13m)p\left(\frac{n-m}{13}\right) \pmod{13}.$$

But since

$$\tau(sp^\lambda) = \tau(p)\tau(sp^{\lambda-1}) - p^{11}\tau(sp^{\lambda-2}), \qquad (s, p) = 1,$$

we have

$$\begin{aligned} \tau(13m) &= \tau(13^\alpha m'), \ (m', 13) = 1, \\ &\equiv \tau(13)\tau(13^{\alpha-1}m') \pmod{13} \\ &\equiv -5\tau(m) \pmod{13}, \end{aligned}$$

and so

$$(12) \qquad p(13n-7) \equiv -2\sum_{m=1}^{n} \tau(m) p\left(\frac{n-m}{13}\right) \pmod{13}.$$

Zuckerman's result gives

$$\sum_{n=0}^{\infty} p(13n+6)x^n = -2\,\frac{f(x^{13})}{f(x)} + 13I$$

from which equation (11) may also be obtained.

3. **Congruence properties, modulus 17.** Let I again denote a power series in x with integral coefficients. As before we express each $\Phi_{r,s}(x)$ in terms of P, Q and R. To each entry must be added $17I$.

TABLE III

$\Phi_{0,1}(x) = -5P + 5$

$\Phi_{0,3}(x) = -8Q + 8$

$\Phi_{0,5}(x) = 3R - 3$

$\Phi_{0,7}(x) = -4Q^2 + 4$

$\Phi_{0,9}(x) = -2QR + 2$

$\Phi_{0,11}(x) = 8Q^3 + 6R^2 + 3$

$\Phi_{0,13}(x) = -5Q^2R + 5$

$\Phi_{1,2}(x) = P^2 - Q$

$\Phi_{1,4}(x) = 3PQ - 3R$

$\Phi_{1,6}(x) = -7PR + 7Q^2$

$\Phi_{1,8}(x) = 3PQ^2 - 3QR$

$\Phi_{1,10}(x) = 4PQR + Q^3 - 5R^2$

$\Phi_{1,12}(x) = 8PQ^3 + 6PR^2 + 3Q^2R$

$\Phi_{1,14}(x) = -3PQ^2R + 8QR^2 - 5$

$\Phi_{2,3}(x) = 3P^3 + 8PQ + 6R$

$\Phi_{2,5}(x) = -3P^2Q + 6PR - 3Q^2$

$\Phi_{2,7}(x) = 3P^2R - 6PQ^2 + 3QR$

$\Phi_{2,9}(x) = -2P^2Q^2 + 4PQR - 3Q^3 + R^2$

$\Phi_{2,11}(x) = -2P^2QR - PQ^3 + 5PR^2 - 2Q^2R$

$\Phi_{2,13}(x) = 3P^2Q^3 - 2P^2R^2 - 2PQ^2R + 8QR^2 - 7$

$\Phi_{2,15}(x) = -8P^2Q^2R + 3PQR^2 + 5P$

$$\Phi_{3,4}(x) = 5P^4 + 4P^2Q + 6PR + 2Q^2$$
$$\Phi_{3,6}(x) = 7P^3Q - 4P^2R + 4PQ^2 - 7QR$$
$$\Phi_{3,8}(x) = 2P^3R - 6P^2Q^2 + 6PQR - Q^3 - R^2$$
$$\Phi_{3,10}(x) = 4P^3Q^2 + 5P^2QR + PQ^3 - 6PR^2 - 4Q^2R$$
$$\Phi_{3,12}(x) = -2P^3QR + 7P^2Q^3 - P^2R^2 - 6PQ^2R + 7QR^2 - 5$$
$$\Phi_{3,14}(x) = -5P^3Q^3 - 8P^3R^2 + P^2Q^2R + 7P - 6PQR^2 - 6Q^3R$$
$$\Phi_{3,16}(x) = -5P^3Q^2R + 6P^2QR^2 + 8P^2 + 4PQ^3R - PR^3 + Q + 4Q^2R^2$$

$$\Phi_{4,5}(x) = -4P^5 + 6P^3Q + 5P^2R - 8PQ^2 + QR$$
$$\Phi_{4,7}(x) = -3P^4Q - 5P^3R - P^2Q^2 - 5PQR + 6Q^3 + 8R^2$$
$$\Phi_{4,9}(x) = -7P^4R - 6P^3Q^2 - 8P^2QR - 3PQ^3 - 3PR^2 - 7Q^2R$$
$$\Phi_{4,11}(x) = -2P^4Q^2 + 8P^3QR - P^2Q^3 + 6P^2R^2 + 8PQ^2R - QR^2 - 1$$

$$\Phi_{5,6}(x) = 4P^6 + 8P^4Q + 7P^3R + 7P^2Q^2 - 6PQR + 3Q^3 - 6R^2$$
$$\Phi_{5,8}(x) = -2P^5Q - 7P^4R - 3P^3Q^3 + 3P^2QR + 3PQ^3 + 4PR^2 + 2Q^2R$$
$$\Phi_{5,10}(x) = -3P^5R - 2P^4Q^2 + 4P^3QR - 2P^2Q^3 - 2P^2R^2 + 2PQ^2R + 2QR^2 + 1$$

$$\Phi_{6,7}(x) = 2P^7 - 8P^5Q + 4P^4R - 6P^3Q^2 - 2P^2QR + 2PQ^3 - 4PR^2 - 5Q^2R$$
$$\Phi_{6,9}(x) = 7P^6Q - 8P^5R + 3P^4Q^2 - 4P^3QR - 6P^2Q^3 - 8P^2R^2 - 8PQ^2R - 4QR^2 - 6$$
$$\Phi_{7,8}(x) = 4P^8 + 7P^6Q + 6P^5R - 7P^4Q^2 - 5P^3QR - P^2Q^3 + 2P^2R^2 + 5PQ^2R + 5QR^2 + 1$$

In the reduction of the powers of Q and R in the expansion of $(Q^3-R^2)^{12}$ we have made use of the relations

$$Q^4 = 3QR^2 - 2 + 17I, \qquad R^3 = 4Q^3R - 3P + 17I,$$

together with combinations of these to form

$$Q^4R = 7PQ - 6R + 17I,$$
$$QR^3 = 8PQ - 7R + 17I,$$
$$Q^3R = 4PQ^2R^2 - 5PQ + 2R + 17I,$$
$$Q^5R^4 = 5P^2Q^2 + 5PQR + 8R^2 + 17I,$$
$$QR^6 = 6Q^3R^2 - 4P^2Q - PR + 17I.$$

We obtain

$$\begin{aligned}\Delta^{12} = (Q^3 - R^2)^{12} = &- P^8 - 4P^6Q + 6P^5R + 3P^4Q^2 \\ &+ 4P^3QR + 6P^2Q^3 - 3P^2R^2 + 2PQ^2R \\ &+ 6QR^2 - 2 + 17I.\end{aligned}$$

Making use of the relations in Table III we have

$$\begin{aligned}\Delta^{12} = &4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x) \\ &- 4\Phi_{3,12}(x) + 2\Phi_{2,13}(x) - 8\Phi_{1,14}(x) + 3P^2R^2 \\ &+ 2QR^2 - 5 + 17I.\end{aligned}$$

But

$$\begin{aligned}3P^2R^2 + 2QR^2 - 5 = &6\Phi_{2,7}(x)\Phi_{0,5}(x) + 6\Phi_{0,9}(x)\Phi_{0,5}(x) + \Phi_{2,7}(x) \\ &- 2\Phi_{1,14}(x) + \Phi_{0,9}(x) + 5\Phi_{0,5}(x) + 17I.\end{aligned}$$

Therefore

$$\begin{aligned}\Delta^{12} = &4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x) - 4\Phi_{3,12}(x) \\ &+ 2\Phi_{2,13}(x) + 7\Phi_{1,14}(x) + \Phi_{2,7}(x) + \Phi_{0,9}(x) + 5\Phi_{0,5}(x) \\ &+ 6\Phi_{2,7}(x)\Phi_{0,5}(x) + 6\Phi_{0,9}(x)\Phi_{0,5}(x) + 17I.\end{aligned}$$

Now we also know that

$$\Delta^{12} = - 4x^{12}[f(x)]^{289}/f(x) + 17I = - 4x^{12}f(x^{289})/f(x) + 17I,$$

and therefore

$$\begin{aligned}x^{12}f(x^{289})/f(x) = &- \Phi_{7,8}(x) + 6\Phi_{6,9}(x) + 8\Phi_{5,10}(x) + 7\Phi_{4,11}(x) \\ &+ \Phi_{3,12}(x) + 8\Phi_{2,13}(x) - 6\Phi_{1,14}(x) + 4\Phi_{2,7}(x) \\ &+ 4\Phi_{0,9}(x) + 3\Phi_{0,5}(x) + 7\Phi_{2,7}(x)\Phi_{0,5}(x) \\ &+ 7\Phi_{0,9}(x)\Phi_{0,5}(x) + 17I.\end{aligned} \tag{13}$$

Equating the coefficients of x^n on both sides of (13) we obtain

$$\begin{aligned}p(n-12) - p(n-301) - \cdots \equiv &- n^7\sigma_1(n) + 6n^6\sigma_3(n) + 8n^5\sigma_5(n) \\ &+ 7n^4\sigma_7(n) + n^3\sigma_9(n) + 8n^2\sigma_{11}(n) - 6n\sigma_{13}(n) + 4n^2\sigma_5(n) \\ &+ 4\sigma_9(n) + 3\sigma_5(n) + 7\sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_5(n-m) \\ &+ 7\sum_{m=1}^{n-1} \sigma_5(m)\sigma_9(n-m) \pmod{17},\end{aligned}$$

where the numbers 12, 301, $\cdots$ are alternately of the form

$(17n-3)(51n-8)/2$ and $(17n+3)(51n+8)/2$. Now replacing n by $17n$ we get

$$p(17n-12)-p(17n-301)-\cdots \equiv 4\sigma_9(17n)+3\sigma_5(17n) + 7\sum_{m=1}^{17n-1} m^2\sigma_5(m)\sigma_5(17n-m) + 7\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m) \pmod{17}.$$

In order to further simplify this result we note that

$$\Phi_{0,5}(x)\Phi_{2,7}(x)-4[\Phi_{1,6}(x)]^2 = -3\Phi_{2,7}(x)+3\Phi_{0,5}(x)\Phi_{0,9}(x)-8\Phi_{0,9}(x) - 6\Phi_{0,5}(x)+17I$$

and therefore

$$\sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_5(n-m) \equiv 4\sum_{m=1}^{n-1} m(n-m)\sigma_5(m)\sigma_5(n-m) - 3n^2\sigma_5(n) + 3\sum_{m=1}^{n-1}\sigma_5(m)\sigma_9(n-m) - 8\sigma_9(n) - 6\sigma_5(n) \pmod{17}.$$

But

$$\sum_{m=1}^{n-1} m(n-m)\sigma_5(m)\sigma_5(n-m) = n\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m) - \sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_5(n-m)$$

and

$$\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m) = \frac{n}{2}\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m)$$

so that finally

$$\sum_{m=1}^{17n-1} m^2\sigma_5(m)\sigma_5(17n-m) \equiv 4\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m) - 5\sigma_9(17n) - 8\sigma_5(17n) \pmod{17}.$$

Therefore

$$p(17n-12)-p(17n-301)-\cdots \equiv \sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m) + 3\sigma_9(17n) - 2\sigma_5(17n) \pmod{17},$$

$$(14)\quad p(17n-12)-p(17n-301)-\cdots \equiv \Sigma_{5,9}(17n) \pmod{17}.$$

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3. H. S. Zuckerman, *Identities analogous to Ramanujan's identities involving the partition function*, Duke Math. J. vol. 5 (1939) p. 89, equation (1.15).

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AUTOMORPHISMS OF FIELDS OF FORMAL POWER SERIES

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We propose to discuss in this note on power series fields in one variable the special automorphisms which do not alter the fields of coefficients. It will be proved that the pseudo-ramification groups introduced by MacLane are universal ramification groups, in the sense that a special ramification group must always be a subgroup of a well determined pseudo-ramification group. Finally we interpret the automorphism group of the field as an automorphism group of an infinite Lie ring.

Let Ω be an arbitrary field of characteristic χ. In the sequel we shall consider the field F of all formal power series $a=\sum_{j>-\infty}^{\infty}\omega_j t^j$ where the ω_j are in Ω and t is a transcendental element over Ω.[1] The field F is complete with respect to the rank one valuation V defined by $Va=m$ where m is the smallest subscript j for which $\omega_j\neq 0$. Let $\mathfrak{O}$ be the valuation ring of all holomorphic series and $\mathfrak{P}=(t)$ the principal prime ideal of $\mathfrak{O}$.

Suppose that S is an automorphism of F. We show that $\mathfrak{O}^S$ is also a valuation ring of F. For the proof[2] let a, b be any two nonzero elements of F. We must show that at least one of the quotients a/b, b/a lies in $\mathfrak{O}^S$. By assumption on S there exist unique elements c, d with $c^S=a$, $d^S=b$. Now observe that at least one of the quotients c/d or d/c lies in $\mathfrak{O}$ for $\mathfrak{O}$ is a valuation ring. Therefore at least one of the

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[1] For the basic properties of valuations see [1, 4, 5, 10]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] See [4, p. 165].