FOURIER SERIES WITH COEFFICIENTS IN A BANACH SPACE

HARRY POLLARD

Let $f(t)$ be a function on $(0, 1)$ to the complex Banach space B. Bochner has shown that the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory [1, pp. 273–276].[1]

Suppose $f(t)$ belongs to L^2 in the sense of Bochner [1]. Define

(1)
$$c_n = \int_0^1 f(t)e^{2\pi i n t}dt.$$

We should expect that the Parseval relation carries over, or at least that the Bessel inequality

(2)
$$\sum \|c_n\|^2 \leq \int_0^1 \|f(t)\|^2 dt$$

is valid. This, however, is not the case; for suitable B we may have $\sum\|c_n\|^2 = \infty$ [1, pp. 275–276].

In this note we detect the root of the trouble by proving that for the validity of (2), B must possess a special character.

THEOREM. *If (2) is valid for all $f(t)$ in L^2 then B is unitary, and conversely.*

B is *unitary* if it admits a *scalar* product with the usual properties [3] (cf. the "normed ring" of Gelfand [2]).

The latter part of the theorem is trivial; we need only apply the classical proof with notational modifications [4, p. 58].

To establish the sufficiency suppose a and b are elements of B. Define
$$f(t) = \begin{matrix} 2a, & (0, 1/2), \\ 2b, & (1/2, 1). \end{matrix}$$

Then

(3)
$$\int_0^1 \|f(t)\|^2 dt = 2[\|a\|^2 + \|b\|^2].$$

By (i) we have

[1] Numbers in brackets refer to the references listed at the end of the paper.

$$c_0 = a + b,$$
$$c_{2n} = 0, \quad n > 0$$
$$c_{2n+1} = (b - a)2/\pi(2n + 1)i.$$

Then

(4) $$\sum \|c_n\|^2 = \|a + b\|^2 + \|a - b\|^2,$$

since

$$\sum_{-\infty}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{4}.$$

By (2), (3), (4)

(5) $$\|a + b\|^2 + \|a - b\|^2 \leqq 2[\|a\|^2 + \|b\|^2].$$

Replace a by $a+b$, b by $a-b$; this simply reverses the inequality of (5). Then for all a, b

$$\|a + b\|^2 + \|a - b\|^2 = 2[\|a\|^2 + \|b\|^2].$$

Hence B is unitary [3].

REFERENCES

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2. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sbornik) N. S. vol. 9 (1941) pp. 3–24.

3. P. Jordan and J. von Neumann, *On inner products in linear, metric spaces*, Ann. of Math. (2) vol. 36 (1935) pp. 719–723.

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KENYON COLLEGE