PROJECTIVE INVARIANTS OF INTERSECTION OF CERTAIN PAIRS OF SURFACES

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1. Introduction. In a recent paper [2][1] the author has shown the existence, together with metric and projective characterizations,[2] of a unique projective invariant determined by the neighborhood of the second order of two surfaces S_1, S_2 at an ordinary point O in ordinary space when the tangent planes τ_1, τ_2 of the surfaces S_1, S_2 at the point O are distinct and the line t of intersection of the two tangent planes τ_1, τ_2 does not coincide with any one of the asymptotic tangents of the surfaces S_1, S_2 at the point O. On the other hand, with regard to the coincidences of the line t and the asymptotic tangents of the surfaces S_1, S_2 at the point O two essentially different cases can arise. The object of this note is to derive some projective invariants in these cases. Noticing that no projective invariant can be determined by the neighborhood of the second order of the surfaces S_1, S_2 at the point O, we obtain all projective invariants determined by the neighborhood of the second order of one surface and that of the third order of the other at the point O.

I. Two surfaces with distinct tangent planes and distinct asymptotic tangents at an ordinary point

2. Derivation of invariants. Let us first consider two surfaces S_1, S_2 in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t is supposed to be an asymptotic tangent of the surface S_1 at the point O. Let t_1 be the other asymptotic tangent of the surface S_1 at the point O, and t_2 the harmonic conjugate line of t with respect to the asymptotic tangents of the surface S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general non-homogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form

$$(1)\qquad S_1:\quad y = lxz + px^3 + rx^2z + sxz^2 + qz^3 + \cdots ,$$

$$(2)\qquad S_2:\quad z = mx^2 + ny^2 + \cdots .$$

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[1] Numbers in square brackets refer to the references at the end of the paper.

[2] An extension of these results to two hypersurfaces has been made by Professor Su. See his paper [6].

In order to find projective invariants of the surfaces S_1, S_2 at the point O, we have to consider the most general projective transformation of coordinates which shall leave the lines t, t_1, t_2 unchanged:

$$
\begin{aligned}
x &= a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
y &= a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
z &= a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),
\end{aligned}
\tag{3}
$$

where a_{ik} are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$
\begin{gathered}
a_{33}l^* = a_{22}a_{44}l, \qquad a_{33}p^* = a_{22}^3 p, \qquad a_{33}q^* = a_{44}^3 q,\\
2a_{12}a_{33}l^* + a_{33}r^* = a_{12}a_{22}a_{44}l + a_{22}^2 a_{44}r,\\
2a_{14}a_{33}l^* + a_{33}s^* = a_{14}a_{22}a_{44}l + a_{22}a_{44}^2 s;\\
a_{44}m^* = a_{22}^2 m, \qquad a_{44}n^* = a_{33}^2 n.
\end{gathered}
\tag{4}
$$

Further elimination of a_{ik} from equations (4) gives immediately that *the quantities*

$$
I = lm/p, \qquad J = p^2q^4m^8/l^{12}n^8 \tag{5}
$$

are projective invariants associated with the surfaces S_1, S_2, at the point O.

3. **Projective characterizations of the invariants I, J.** It is well known that the tangent plane τ_1 intersects the surface S_1 in a curve with a node at the point O, the nodal tangents being the asymptotic tangents t, t_1. The expansion of the branch C_1 of this curve which is tangent to the tangent t, $z=y=0$, is easily found to be

$$
z = -(p/l)x^2 + \cdots . \tag{6}
$$

On the other hand, the curve C_2 in which the tangent plane τ_1 intersects the surface S_2 is given by the equations

$$
z = mx^2 + \cdots, \qquad y = 0. \tag{7}
$$

It is thus easy to reach the conclusion that *the projective invariant I associated with the point O of the surfaces S_1, S_2 is, except for sign, equal to the projective invariant of contact*[3] *of the two plane curves C_1, C_2 at the point O.*

To characterize projectively the other invariant J it is useful to

[3] The projective invariant of contact of two plane curves having ordinary contact at a point was first found by H. J. S. Smith [5] and R. Mehmke [3]; and its simple projective characterization was later given by C. Segre [4].

consider the asymptotic curves Γ, Γ_1 of the surface S_1 at the point O. The expansions of these asymptotic curves at the point O are found, after a simple calculation, to be

$$(8) \qquad \Gamma: \quad y = -2px^3 + \cdots, \qquad z = -(3p/l)x^2 + \cdots;$$

$$(9) \qquad \Gamma_1: \quad y = -2qz^3 + \cdots, \qquad x = -(3q/l)z^2 + \cdots.$$

Let K be any five-point quadric cone of the asymptotic curve Γ at the point O with O for vertex and with an asymptotic tangent of the surface S_2 at the point O for a generator, then making use of equations (8) we can easily obtain its equation, namely,

$$(10) \qquad z^2 + (9p/2l^2)xy \pm (9p/2l^2)(-n/m)^{1/2}y^2 + kxz = 0,$$

where k is arbitrary. Similarly, we have a five-point quadric cone K_1 of the asymptotic curve Γ_1 at the point O such that it has O for vertex and the line t_2 for a generator and is tangent to the plane τ_2, $z=0$, along t_2. The equation of this cone K_1 is

$$(11) \qquad x^2 + (9q/2l^2)yz = 0.$$

From equations (10), (11) it follows that the cone projecting the curve of intersection of the cones K, K_1 from the line t_2 consists of the four planes:

$$(12) \qquad 9q^2z^4 - 18kl^2qx^2z^2 - 9pqx^3z \pm 2l^2p(-n/m)^{1/2}x^4 = 0,$$

which are intersected by the plane τ_1 in four lines λ_i $(i=1, \cdots, 4)$ with equations

$$(13) \qquad z + \alpha_i x = 0, \qquad y = 0 \qquad (i = 1, \cdots, 4),$$

where the coefficients α_i satisfy the relation

$$(14) \qquad \alpha_1\alpha_2\alpha_3\alpha_4 = \pm (2l^2p/9q^2)(-n/m)^{1/2}.$$

On the other hand, associated with the point O of the surface S_1 there are three tangents of Darboux, whose equations may be found by making use of equations (8), (9) and Bompiani's result [1] that they are the three principal lines of the asymptotic curves Γ, Γ_1 at the point O. The result is

$$(15) \qquad px^3 + qz^3 = 0, \qquad y = 0.$$

Let d be any one of the tangents (15) of Darboux, and let D_i denote the cross ratio of the four lines t, t_1, d, λ_i $(i=1, \cdots, 4)$; then

$$(16) \qquad D_i = (tt_1, d\lambda_i) = (1/\alpha_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4).$$

A reference to equations (5), (14), (16) suffices to substantiate that *the invariant J can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows:*

$$J = -(2/9)^6(D_1D_2D_3D_4)^6. \tag{17}$$

II. Two surfaces with distinct tangent planes and a common asymptotic tangent at an ordinary point

4. Derivation of an invariant. Finally, suppose that S_1, S_2 be two surfaces in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t stands for one asymptotic tangent of both surfaces S_1, S_2 at the point O; and that t_1, t_2 be respectively the other asymptotic tangents of the surfaces S_1, S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form (1) and

$$S_2: \quad z = mxy + \cdots . \tag{18}$$

In a way similar to the foregoing we can easily show that *the quantity*

$$I = p^4q^2/l^9m^3 \tag{19}$$

is a projective invariant associated with the surfaces S_1, S_2 at the point O.

5. A projective characterization of the invariant I. Let Γ, Γ_1 be the asymptotic curves of the surface S_1 at the point O whose tangents are t, t_1 respectively. Among the four-point quadrics of the asymptotic curve Γ at the point O we can determine a two-parameter family such that every one of the family has t_2 for a generator and has at the point O contact of the second order with the surface S_2. By means of equations (8), (18) it is easy to obtain the equation of a general quadric of this family, namely,

$$z + (3p/l)x^2 - mxy + Eyz + Fz^2 = 0, \tag{20}$$

where E, F are arbitrary. The quadric (20) is cut by the plane τ_2 in the line t_2 and a line with equations

$$3px - lmy = 0, \qquad z = 0. \tag{21}$$

If any five-point quadric cone, with the point O for vertex, of the asymptotic curve Γ at O passes through the line (21), then it must have the equation:

$$z^2 + (9p/2l^2)xy - (3m/2l)y^2 + kyz = 0, \tag{22}$$

where k is arbitrary. The cone projecting the curve of intersection of the cones (11), (22) from the line t_2 consists of the four planes:

$$27q^2z^4 - 6kl^2qx^2z^2 - 27pqx^3z - 2l^3mx^4 = 0, \tag{23}$$

which are intersected by the plane τ_1 in four lines μ_i $(i=1, \cdots, 4)$:

$$z + \beta_i x = 0, \qquad y = 0 \qquad (i = 1, \cdots, 4), \tag{24}$$

where the coefficients β_i satisfy the relation

$$\beta_1\beta_2\beta_3\beta_4 = -2l^3m/27q^2. \tag{25}$$

If d be any one of the three tangents of Darboux associated with the point O of the surface S_1 and if D_i denotes the cross ratio of the four lines t, t_1, d, μ_i $(i=1, \cdots, 4)$, then from equations (15), (24) it follows that

$$D_i = (tt_1, d\mu_i) = (1/\beta_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4). \tag{26}$$

A reference to equations (19), (25), (26) suffices to show that *the invariant I can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows*:

$$I = -(2/27)^3(D_1D_2D_3D_4)^3. \tag{27}$$

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