ON RIESZ SUMMABILITY OF FOURIER SERIES BY EXPONENTIAL MEANS

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Let $f(t)$ be an integrable periodic function with the period 2π. Let its Fourier series be

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{1}$$

and let

$$\phi(t) = \{f(x+t) + f(x-t) - 2s\}/2,$$

$$\phi_\beta(t) = (1/\Gamma(\beta)) \int_0^t (t-u)^{\beta-1}\phi(u)du,$$

$$A_n = a_n \cos nx + b_n \sin nx.$$

We shall prove the following result.[1]

If $A_n > -Kn^{-\beta/\gamma}$ $(\gamma > \beta > 0)$ *and*

$$\phi_\beta(t) = o(t^\gamma) \qquad \text{as } t \to 0, \tag{2}$$

the Fourier series (1) *converges to* s *at* $t = x$.

Set $\alpha = 1 - \beta/\gamma$, and

$$C_\tau(\omega) = a_0 e^{\tau\omega^\alpha}/2 + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^\tau A_n. \tag{3}$$

The Fourier series (1) is said to be summable (e^{n^α}, τ) to the sum s if[2]

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty.$$

Concerning this kind of summability we have the following theorem.

THEOREM.[3] *If* (2) *holds and* τ *is a positive integer greater than* $\gamma+1$ *the Fourier series* (1) *is summable* (e^{n^α}, τ) *to the sum* s *at* $t = x$.

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[1] G. H. Hardy and J. E. Littlewood [2], F. T. Wang [6]. Numbers in brackets refer to the references listed at the end of the paper.

[2] G. H. Hardy and M. Riesz [3].

[3] Under the hypotheses of the Theorem I have established that the Fourier series (1) is summable $(e^{n^\alpha}, \gamma+\delta)(\delta>0)$ to the sum s at $t=x$, but the proof is very complicated. See F. T. Wang [6].

The convergence criterion above is deducible from this theorem by the use of a result of Hardy,[4] namely *if the series* $\sum_{n=0}^{\infty} a_n$, *with terms* $a_n \geqq -Kn^{\alpha-1}$, $0<\alpha<1$, *is summable* (e^{n^α}, τ), *it is convergent.*

To prove the theorem we write

$$E_\tau(\omega, t) = \tau\alpha \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx.$$

Then we have[5] $C_\tau(\omega) = (2/\pi)\int_0^1 \phi(t) E_\tau(\omega, t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha})$, and if we take $\omega_1 = 2^{-1/\alpha}\omega$, then

$$\begin{aligned} E_\tau(\omega, t) &= \alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx + o(e^{\tau\omega^\alpha}) \\ &= F_\omega(t) + o(e^{\tau\omega^\alpha}). \end{aligned}$$

Hence

$$C_\tau(\omega) = (2/\pi) \int_0^1 \phi(t) F_\omega(t) dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{4}$$

By setting $n = [\beta]+1$ and differentiating under the integral sign we get

$$\begin{aligned} F_\omega^{(n)}(t) &= \frac{d^n}{dt^n}\left\{\alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx\right\} \\ &= \sum_{i=0}^n K_i \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i} \frac{\sin (xt-a)}{t^{i+1}} dx, \end{aligned} \tag{5}$$

(5)[6]

where $a = (n-i)\pi/2$.

By mathematical induction we can easily establish the formula

$$\begin{aligned} &(d^r/dx^r)\left\{(e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i}\right\} \\ &\qquad = \sum_{j=1}^{\tau} \sum_{p=0}^{r} K_{ip} e^{(\tau-j)\omega^\alpha} e^{jx^\alpha} x^{(p+1)(\alpha-1)+n-i-(r-p)}. \end{aligned} \tag{6}$$

Then by the use of (5) and (6) and an integration by parts we find

$$\begin{aligned} F_\omega^{(n)}(t) = &\sum_{i=0}^{n} \sum_{j=1}^{\tau} \sum_{p=c}^{\tau-1} K_{ijp} e^{(\tau-j)\omega^\alpha} \int_{\omega_1}^{\omega} e^{jx^\alpha} x^c \frac{\sin (xt-b)}{t^{i+\tau}} dx \\ &+ O(e^{(\tau-1/2)\omega^\alpha} \omega^{k_1} t^{-k_2}), \end{aligned} \tag{7}$$

[4] G. H. Hardy [4].

[5] F. T. Wang [6].

[6] Throughout this paper we use K or $K_i \cdots$ as a constant different in different occurrences.

where $c=(p+1)(\alpha-1)+n-i-(\tau-1-p)$, and $b=(n-i-\tau+1)\pi/2$.

Put $t=1$ in (7). Then $F_\omega^{(m)}(0)$ is finite for $1\leqq m<n$, and $F_\omega^{(m)}(1) = o(e^{\tau\omega^\alpha})$. Successive integration of (4) by parts yields

$$(8) \qquad C_\tau(\omega) = (2/\pi)\int_0^1 \phi_n(t) F_\omega^{(n)}(t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}).$$

By a theorem on the fractional integral[7]

$$\phi_n(t) = (1/\Gamma(n-\beta))\int_0^t (t-u)^{n-\beta-1}\phi_\beta(u)du,$$

we have

$$(9) \qquad C_\tau(\omega) = (2/\pi)\int_0^1 \phi_\beta(u)H_\omega(u)du + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}),$$

where

$$(10) \qquad \begin{aligned} H_\omega(u) &= (1/\Gamma(n-\beta))\int_u^1 (t-u)^{n-\beta-1}F_\omega^{(n)}(t)dt \quad &(n>\beta>n-1) \\ &= F_\omega^{(n)}(u) &(n=\beta). \end{aligned}$$

Concerning $H_\omega(u)$ we require the following two lemmas.

LEMMA 1. *For* $\omega>K$ *and* $0<u<1$,

$$\begin{aligned} H_\omega(u) = &\sum_{i=0}^{n-1} O(e^{\tau\omega^\alpha}\omega^{\beta-i}u^{-i-1}) + \sum_{i=0}^{n-1} O(e^{\tau\omega^\alpha}\omega^{n-i-1}u^{n-\beta-i-2}) \\ &+ O(e^{\tau\omega^\alpha}\omega^{n-1}(1-u)^{n-\beta-1}) + O(e^{\tau\omega^\alpha}u^{-\beta-1}). \end{aligned}$$

PROOF. From (10) and (5) we have

$$(11) \qquad \begin{aligned} H_\omega(u) = &\sum_{i=0}^{n} K_i \int_{\omega_1}^{\omega} (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1}e^{x^\alpha}x^{\alpha-1+n-i}dx \\ &\cdot \int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt. \end{aligned}$$

Now

$$(12) \qquad \int_1^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt = O\{(1-u)^{n-\beta-1}x^{-1}\}.$$

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[7] L. S. Bosanquet [1].

It follows from a change of variable, the second mean value theorem, and a theorem on the Γ function,[8] that

$$
\begin{aligned}
&\int_u^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}\,dt\\
(13)\qquad &= u^{n-\beta-i-1}\int_0^\infty v^{n-\beta-1}\sin(x^u(1+v)-a)dv + O(u^{n-\beta-i-2}x^{-1})\\
&= O(u^{-i-1}x^{\beta-n}) + O(u^{n-\beta-i-2}x^{-1}).
\end{aligned}
$$

The lemma is proved by (11), (12), (13) and an easy estimate of the term $i=n$ in (11).

LEMMA 2. *For* $\omega>K$ *and* $0<u<1$,

$$
\begin{aligned}
H_\omega(u) = &\sum_{i=0}^{n} O(e^{\tau\omega^\alpha}u^{-\tau-i-1}\omega^{\tau(\alpha-1)+\beta-i}) + O(e^{(\tau-1/2)\omega^\alpha}\omega^{k_1}u^{-k_2})\\
&+\sum_{i=0}^{n} O(e^{\tau\omega^\alpha}u^{n-\beta-\tau-i-1}\omega^{(\tau-1)(\alpha-1)+n-i-1})\\
&+O(e^{\tau\omega^\alpha}\omega^{(\tau-1)(\alpha-1)+n-1}(1-u)^{n-\beta-1}).
\end{aligned}
$$

PROOF. From (13) we get

$$
\begin{aligned}
H_\omega(u) = &\sum_{i=0}^{n}\sum_{j=1}^{\tau}\sum_{p=0}^{\tau-1} K_{ijp}e^{(\tau-j)\omega^\alpha}\int_{\omega_1}^{\omega} e^{ix^\alpha}x^c dx\\
(14)\qquad &\cdot\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt + O(e^{(\tau-1/2)\omega^\alpha}\omega^{k_1}u^{-k_2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt\\
&= u^{n-\beta-i-\tau}\int_0^\infty v^{n-\beta-1}\sin\{xu(1+v)-b\}dv\\
(15)\qquad &\quad + O((1-u)^{n-\beta-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1})\\
&= u^{-\tau-i}x^{\beta-n}\Gamma(\beta-n)\sin(xu-b')\\
&\quad + O((1-u)^{n-p-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1}).
\end{aligned}
$$

From (14) and (15), Lemma 2 follows.

[8] E. C. Titchmarsh [5, p. 107].

PROOF OF THE THEOREM. By Lemma 1 and (2)

$$\int_0^{\omega^{\alpha-1}} \phi_\beta(u)H_\omega(u)du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty \tag{16}$$

and by (2) and Lemma 2

$$\int_{\omega^{\alpha-1}}^{1} \phi_\beta(u)H_\omega(u)du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty \tag{17}$$

By (9), (16), and (17), then,

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty$$

Thus the theorem is proved.

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