A NOTE ON RIESZ SUMMABILITY OF THE TYPE e^{n^α}

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Recently I proved the following result in the case $r=2$ (Wang [4][1]).

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^\infty a_n$. If $\sigma_n^{(r)}-s=o(n^{-r\alpha})$ $0<\alpha<1$, as $n\to\infty$, where r is a positive integer, and $a_n>-Kn^{\alpha-1}$, the series converges to sum s.

For the case $r=1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type e^{n^α}, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_\tau(\omega)=a_0e^{\tau\omega^\alpha}+\sum_{n<\omega}(e^{\omega^\alpha}-e^{n^\alpha})^\tau a_n$. A series $\sum_{n=0}^\infty a_n$ is said to be summable (e^{n^α},τ) to the sum s if

$$C_\tau(\omega) = se^{\tau\omega^\alpha}+o(e^{\tau\omega^\alpha}). \tag{1}$$

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^\infty a_n$, with terms $a_n\geqq -Kn^{\alpha-1}$, $0<\alpha<1$, is summable (e^{n^α},τ), it is convergent. We shall now prove the following theorem.

THEOREM. *If $\sigma_n^{(r)}-s=o(n^{-r\alpha})$, $0<\alpha<1$, as $n\to\infty$, the series $\sum_{n=0}^\infty a_n$ is summable (e^{n^α},τ) to the sum s, where $\tau>r/(1-\alpha)$.*

To prove this let $\beta_n=(e^{\omega^\alpha}-e^{n^\alpha})^\tau$, $\Delta\beta_n=\beta_n-\beta_{n+1}$, $\Delta^{r+1}\beta_n=\Delta^r\beta_n-\Delta^r\beta_{n+1}$ and

$$s_n^{(r)}=\sum_{\nu=0}^n\binom{n-\nu+r}{n-\nu}a_\nu,$$

$m=[\omega]$. Then, by successive Abel's transformations we have

$$\begin{aligned} C_\tau(\omega) &= a_0e^{\tau\omega^\alpha}+\sum_{n=1}^m\beta_na_n \\ &= a_0e^{\tau\omega^\alpha}+\sum_{n=1}^{m-r+1}s_n^{(r)}\Delta^{r+1}\beta_n+\sum_{i=0}^r s_{m-i}^{(i)}\Delta^i\beta_{m-i}-\sum_{i=0}^r s_0^{(i)}\Delta^i\beta_1 \qquad (2)\\ &= a_0e^{\tau\omega^\alpha}+J_1+J_2-J_3. \end{aligned}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.

Since $\beta_{m-i}=(e^{\omega^\alpha}-e^{(m-i)^\alpha})^\tau=O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)})$, it follows that

$$\Delta^{(i)}\beta_{m-i} = O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)}) \qquad \text{for } 0 \leqq i \leqq r.$$

By a familiar theorem on Cesàro sums we get

$$s_{m-i}^{(i)} = O(\omega^r), \qquad \text{for } 0 \leqq i \leqq r,$$

and from this

$$J_2 = O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)+r}) = o(e^{\tau\omega^\alpha}). \tag{3}$$

Since $\Delta^i\beta_1=O(e^{\tau\omega^\alpha}\omega^{i(\alpha-1)})$, for $1\leqq i\leqq r$, and $\beta_1=(e^{\omega^\alpha}-e)^\tau$, $s_0^{(i)}=s_0=a_0$ we get

$$J_3 = e^{\tau\omega^\alpha}a_0 + o(e^{\tau\omega^\alpha}). \tag{4}$$

By the hypothesis of the theorem we have

$$\begin{aligned} J_1 = \sum_{n=1}^{m-r-1} s_n^{(r)}\Delta^{r+1}\beta_n &= s\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n \\ &+ o\left(\sum_{n=1}^{m-r-1} n^{r(1-\alpha)}\left|\Delta^{r+1}\beta_n\right|\right). \end{aligned} \tag{5}$$

It follows by mathematical induction that

$$\Delta^{r+1}\beta_n = (-1)^{r+1}\int_n^{n+1}dx_1\int_{x_1}^{x_1+1}dx_2\cdots\int_{x_r}^{x_r+1}B^{(r+1)}(x_{r+1})dx_{r+1},$$

where

$$\beta^{(r+1)}(x) = \frac{d^{r+1}}{dx^{r+1}}\left\{(e^{\omega^\alpha}-e^{x^\alpha})^\tau\right\}.$$

By direct differentiation we have

$$B^{(r+1)}(x) = \sum_{j=1}^{r+1}\sum_{i=1}^{\tau} c_{ij}e^{(\tau-j)\omega^\alpha}e^{jx^\alpha}x^{i\alpha-r-1},$$

where τ is taken as a positive integer and the constants c_{ij} depend upon i, j, τ, r, α. Hence we get

$$\Delta^{r+1}\beta_n = \sum_{i=1}^{\tau} O(e^{(\tau-i)\omega^\alpha}e^{in^\alpha}n^{(r+1)(\alpha-1)}). \tag{6}$$

It is easily verified by Abel's transformation that

$$\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n = e^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{7}$$

Hence by (5), (6), and (7)

$$J_1 = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) + o\left(\sum_{i=1}^{\tau}\sum_{n=1}^{m-r-1} e^{(\tau-i)\omega^\alpha}e^{in^\alpha}n^{\alpha-1}\right) \tag{8}$$
$$= se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}).$$

The proof of the theorem follows from (4), (2), (3), and (8).

I conclude by observing that the theorem is the best possible of its kind (Wang [4]).

References

1. R. P. Boas, *Tauberian theorem for* (C.1) *summability*. Duke Math. J. vol. 4 (1938) pp. 227–230.
2. G. H. Hardy, *An extension of a theorem on oscillating series*, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 174–180.
3. G. H. Hardy and M. Riesz, *The general theory of Dirichlet series*, 1914.
4. F. T. Wang, *Some remarks of oscillating series*, Quarterly J. Math.

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