CLUSTER POINTS OF SUBSEQUENCES[1]

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In the preceding paper [1][2] Buck defines a class of "subsequences" of a multiple sequence and shows that "almost all" of such subsequences have certain properties. This note is essentially based on a different choice of the definition of "subsequences"; that is, this paper and [1] are generalizations in different directions of a preceding paper by Buck and Pollard (reference 2 of [1]). In this discussion countability is the important property of the index systems such as the integers underlying the simple sequences or the n-tuples of integers underlying the multiple sequences. Countability is a slightly stronger condition than is necessary since the results will be shown to hold as well for functions of n variables as for multiple sequences; some other special cases are mentioned at the end of this paper. Also I modify Buck's approach by considering cluster points in neighborhood spaces rather than limit points in convergence spaces [3]. It may be mentioned that even for multiple sequences Theorems 1 and 2 of these papers are independent since Buck's set of "subsequences" is a set of measure zero in the set of "subsequences" considered here; my Theorem 3 contains the corresponding theorem of [1] as a special case. Lemma 1 and its corollary, Lemma 3, are the fundamental results on which the theorems rest; Lemma 3 is the generalization appropriate to this paper of the lemma in §3 of [1].

1. **Preliminaries.** If R is any set, a product measure can be defined in the set of characteristic functions of subsets of R [1, footnote 2] and this in turn induces a measure $|\cdots|$ for subsets of the set $\mathcal{E}$ of all subsets E of R; this measure is non-negative, completely additive, and (if R is infinite) takes all values between 0 and 1 inclusive; its other principal characteristic is that if $r_1, \cdots, r_k \in R$, then[3] $\{E \mid \text{no } r_i \in E\}$ is of measure 2^{-k}; hence if E_0 is an infinite subset of R and $A = \{E \mid E \cap E_0 \text{ is empty}\}$, $|A| = 0$.

An *index system* $\mathfrak{R} = (R, \geqq)$ is a set R and a binary relation $\geqq$ such that $\geqq$ is transitive and every element r_0 has a successor $r_1 > r_0$ such that $r_0 \not> r_1$. (In the language of [4] $\mathfrak{R}$ is oriented and has no terminal

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[1] Considerations suggested by the preceding paper of R. C. Buck.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] The usual notation of $\cup$ and $\cap$ will be used for union and intersection of sets; $\{p \mid P\}$ will mean the set of all p having the property P.

elements.) A set E in $\mathcal{R}$ is called *cofinal* in $\mathcal{R}$ if for every r in R there exists $r' \geqq r$ with r' in E. Let $E^* = \{r \mid r \geqq \text{some } r' \text{ in } E\}$.

Note that if $\mathcal{R}$ is the system of integers ordered by magnitude, then the cofinal subsets of $\mathcal{R}$ are the infinite subsets; for g defined on a general index system $\mathcal{R}$ it is clear that reducing the domain of definition of g to a cofinal subset E of $\mathcal{R}$ is a generalization of the process of selecting a subsequence in case $\mathcal{R}$ is the system of integers.

A subsystem $\mathcal{R}' = (R', \geqq)$ of $\mathcal{R} = (R, \geqq)$ is a subset R' of R with the order relation between points of R' defined by that in R; if R' is cofinal in the index system $\mathcal{R}$, then $(R', \geqq)$ is also an index system. Cofinality is transitive in a transitive system; that is, if R' is cofinal in $(R, \geqq)$ and R'' is cofinal in $(R', \geqq)$, then R'' is cofinal in $(R, \geqq)$.

We may note that if $\mathcal{R}$ is the set of n-tuples of integers (the case studied in [1]), where $(i_1, \cdots, i_n) \geqq (j_1, \cdots, j_n)$ if and only if $i_k \geqq j_k$ for every $k \leqq n$, then the product subsets defined by Buck are cofinal in $\mathcal{R}$ but are very sparsely distributed in the set of all cofinal subsets of $\mathcal{R}$; to be precise, such sets form a set of measure 0 if $n \geqq 2$. A product set in the set of n-tuples of integers, $R = I \times I \times \cdots \times I$, is a set of the form $E_1 \times E_2 \times \cdots \times E_n$, $E_i \subset I$; these product sets define the class of "subsequences" used by Buck. If E_k is the set of elements of R with all coordinates not greater than k and if A_k is the class of all subsets E of R such that $E \cap E_k$ is a product set in E_k, then $A = \bigcap_k A_k$. It is easily seen that if $E' \subset E_k$, $\{E \mid E \cap E_k = E'\}$ is of measure 2^{-k^n}; since there are $(2^k - 1)^n + 1 < 2^{nk}$ product sets in E_k, it follows that $|A_k| < 2^{nk - k^n}$; if $n \geqq 2$, this tends to zero as k increases so $|A| = 0$.

LEMMA 1. *If $\mathcal{R}$ has a countable cofinal subset and $\mathcal{C}$ is the set of all cofinal subsets of $\mathcal{R}$, then $|\mathcal{C}| = 1$.*

Let R' be a countable cofinal subset of $\mathcal{R}$ and suppose $E \notin \mathcal{C}$; then there exists r in R such that $(r)^* \cap E$ is empty. Since there exists r' in R' such that $r' \geqq r$, it follows that $(r')^* \cap E$ is empty. Since r' has an infinite number of distinct successors in R', the set $A_{r'} = \{E \mid (r')^* \cap E$ is empty$\}$ is of measure zero. Since $\mathcal{E} - \mathcal{C} = \bigcup_{r' \in R'} A_{r'}$, $|\mathcal{E} - \mathcal{C}| = 0$ so $|\mathcal{C}| = 1$.

By means of this lemma we can define a measure in $\mathcal{C}$ by taking the measure in $\mathcal{E}$ of elements of $\mathcal{C}$; since $|\mathcal{C}| = 1$, we can talk meaningfully about almost all cofinal subsets of $\mathcal{R}$ [5, Theorem 1.1]. Note that cofinality of E is not affected by adding or removing a finite set, so $\mathcal{C}$ is a "homogeneous" subset of $\mathcal{E}$ and therefore if it is measurable must have measure 0 or 1; which case occurs when $\mathcal{R}$ does not have a countable cofinal subset, I do not know.

X is a *neighborhood space* [3] if for each x in X is defined a nonempty family of subsets of X, the neighborhoods of x. If g is a function defined on an index system $\mathcal{R}$ with values in a neighborhood space X, x is a *limit point* of g (symbol: $x=\lim_{(R,\geqq)} g$) if for each neighborhood N of x and every r_0 in R there exists $r_1 \geqq r_0$ such that $g(r) \in N$ whenever $r \geqq r_1$. (This definition is due to Alaoglu and Birkhoff [2]; in case $\mathcal{R}$ is directed it reduces to the standard simpler form: $x=\lim_{(R,\geqq)} g$ if for each N there exists r_N in R such that $g(r) \in N$ whenever $r \geqq r_N$. $\mathcal{R}$ is directed if every pair of elements has a common successor.) A point x is called a *cluster point* of g if for every neighborhood N of x and every r_0 in R there exists $r_1 \geqq r_0$ such that $g(r_1) \in N$. Clearly every limit point of g is a cluster point of g, but not conversely. (See Lemma 2 below.) If g is a function from $\mathcal{R}$ into X and E is a cofinal subset of $\mathcal{R}$, let g_E be the function g reduced onto E and let Qg_E be the set of cluster points of g_E; the function g_E and the set Qg_E will play a role here analogous to that played by the subsequence x' and the set Px' in [1]. Clearly $x=\lim_{(R,\geqq)} g$ implies $x=\lim_{(E,\geqq)} g_E$ for every E cofinal in $\mathcal{R}$. Let Pg_E be the set of limit points of functions $g_{E'}$ for E' cofinal in E; that is, $x \in Pg_E$ if and only if there exists E' cofinal in $(E, \geqq)$ such that $x=\lim_{(E',\geqq)} g_{E'}$.

Recall that X is said to satisfy Hausdorff's *first countability condition* if for each x in X there is a countable set $\{N_i\}$ of neighborhoods of x such that each neighborhood of x contains an N_i. The next lemma shows the connection between Qg and Pg.

LEMMA 2. *If $\mathcal{R}$ has a countable cofinal subsystem, if X satisfies the first countability condition, and if the intersection of each pair of neighborhoods of each point x of X contains a third neighborhood of x, then x is a cluster point of g if and only if there exists E cofinal in $\mathcal{R}$ such that $x=\lim_{(E,\geqq)} g_E$; that is, $Qg=Pg$.*

$Qg \supset Pg$ with no restriction on $\mathcal{R}$ or X, for $x=\lim_{(E,\geqq)} g_E$ and N a neighborhood of x imply that if $r \in R$, there exists r_1 in E with $r_1 \geqq r$ and then an r_2 in E such that $r_2 \geqq r_1$ and $g(r_2) \in N$. If $\mathcal{R}$ and X are restricted as above and if x is a cluster point of g, there exists a sequence $\{N_i'\}$ of neighborhoods of x such that each neighborhood of x contains an N_i'. By the other condition there exists a decreasing sequence of such neighborhoods $N_1 \supset N_2 \supset \cdots \supset N_i \supset \cdots$. Enumerate R in a sequence $\{r_j'\}$; then let r_1 be a point of $g^{-1}(N_1)$ which follows r_1'; let r_2 and r_3 be points of $g^{-1}(N_2)$ which, respectively, follow r_1 and r_2'; let r_4, r_5, r_6 be points of $g^{-1}(N_3)$ which follow r_2, r_3, and r_3', and so on. Then $E=\{r_i\}$ contains a successor of every element of R', so is cofinal in $(R', \geqq)$ and hence cofinal in $\mathcal{R}$. If N is a neighborhood of x, there is an $N_j \subset N$ and there exists n such that $g(r_i) \in N_j$ if $i \geqq n$.

Since the set of all r which do not precede any r_i, $i<n$, is cofinal in R and contains all successors of each of its elements, its intersection with E is a set of the same sort in E; this shows that E has the desired property; that is, that $x=\lim_{(E,\geqq)} g_E$.

Note that no such relation holds for multiple sequences if the cofinal sets of $\mathcal{R}$ which are used are restricted as in [1] to be product sets.

We used Lemma 1 to show that "almost everywhere" has meaning in $\mathcal{C}$; a simple application of the same proof gives the next result which can be regarded as an extension of the lemma of [1, §3].

LEMMA 3. *If $\mathcal{R}$ has a countable cofinal subsystem, if E_0 is cofinal in $\mathcal{R}$, and if $A=\{E \mid E\cap E_0$ is not cofinal in $E_0\}$, then $|A|=0$; that is, almost every E of $\mathcal{C}$ meets E_0 in a set cofinal in $\mathcal{R}$.*

Let E_1 be a countable subset of E_0 cofinal in $(E_0, \geqq)$; then $E\cap E_0$ not cofinal in E_0 means that there exists r_E in E_1 such that $E\cap E_0\cap(r_E)^*$ is empty. For fixed r in E_1 let $A_r=\{E \mid E\cap E_0\cap(r)^*$ is empty$\}$; since $E_0\cap(r)^*$ is infinite, $|A_r|=0$; since $A=\bigcup_{r\in E_1}A_r$, $|A|=0$.

2. **Cluster points.** We now proceed to the analogues of the theorems of [1].

THEOREM 1. *If $\mathcal{R}$ is an index system with a countable cofinal subset, if X satisfies the first countability condition, if g is a function from $\mathcal{R}$ into X, and if $x\in Qg$, then $x\in Qg_E$ for almost every E of $\mathcal{C}$; that is, each cluster point of g is a cluster point of almost every g_E.*

x is a cluster point of g if and only if $g^{-1}(N)$ is cofinal in $\mathcal{R}$ for every neighborhood N of x. If $\{N_i\}$ is an equivalent sequence of neighborhoods of x, let $A_i=\{E \mid E\cap g^{-1}(N_i)$ is not cofinal in $g^{-1}(N_i)\}$; then, by Lemma 3, $|A_i|=0$. Setting $A=\mathcal{C}-\bigcup_i A_i$, $|A|=|\mathcal{C}|=1$. If $E\in A$ and N is a neighborhood of x, there is an $N_i\subset N$; since $E\notin A_i$, $E\cap g^{-1}(N_i)$ is cofinal in $g^{-1}(N_i)$ and hence cofinal in $\mathcal{R}$. Since $E\cap g^{-1}(N)\supset E\cap g^{-1}(N_i)$, $E\cap g^{-1}(N)$ is also cofinal in $\mathcal{R}$ and therefore is cofinal in $(E, \geqq)$; that is, if N is a neighborhood of x and $E\in A$, $g^{-1}(N)\cap E$ is cofinal in $(E, \geqq)$, that is, x is a cluster point of g_E if $E\in A$.

Limit points have an analogous property.

THEOREM 1'. *If $\mathcal{R}$ and X satisfy the hypotheses of Theorem 1, then $x=\lim_{(R,\geqq)} g$ if and only if $x=\lim_{(E,\geqq)} g_E$ for almost every E in $\mathcal{C}$.*

If $x=\lim_{(R,\geqq)} g$, then $x=\lim_{(E,\geqq)} g_E$ for every E in $\mathcal{C}$. If $x\neq\lim_{(R,\geqq)} g$, there exists a neighborhood N of x and an r_0 in $\mathfrak{R}$ such that every $r_1 > r_0$ has a successor $r_2 > r_1$ for which $g(r_2) \notin N$; let $E_0 = \{r \mid g(r) \in X - N \text{ and } r > r_0\}$; then if E_1 is so chosen that E_0 and E_1 have no common successors and $E_0 \cup E_1$ is cofinal in $\mathfrak{R}$, by Lemma 3 the set $A = \{E \mid E \cap (E_0 \cup E_1) \text{ is cofinal in } \mathfrak{R}\}$ is of measure 1. For any such E, $E \cap E_0$ is cofinal in $(E_0, \geqq)$ so $x \neq \lim_{(E,\geqq)} g_E$ if $E \in A$; that is, $x \neq \lim_{(R,\geqq)} g$ implies $x \neq \lim_{(E,\geqq)} g_E$ for almost every E in $\mathfrak{C}$.

Say that g is *divergent* if $x = \lim_{(R,\geqq)} g$ is false for every x in X.

COROLLARY. *Let X and $\mathfrak{R}$ satisfy the conditions of the theorem and suppose that g is divergent; then for each x in X the set $A_x = \{E \mid x = \lim_{(E,\geqq)} g_E\}$ is of measure zero. Hence if almost every g_E has a limit point, then Pg is uncountable.*

The first statement follows immediately from the theorem. For the second, $\{E \mid g_E \text{ has a limit point}\} = \bigcup_{x \in Pg} A_x$; since $|A_x| = 0$ and $|\bigcup_{x \in Pg} A_x| = 1$, Pg is uncountable.

The next two results are related to Theorem 1′ but stronger hypotheses enable us to draw stronger conclusions.

THEOREM 2. *If X and $\mathfrak{R}$ satisfy the conditions of Theorem 1 and if each pair of distinct points of X has a pair of disjoint neighborhoods,*[4] *then g is divergent if and only if almost every g_E is divergent.*

If g has the limit x, so does every g_E. If g is divergent, by the corollary $|A_x| = 0$ for every x. By the first statement in the proof of Lemma 2, if $x_1 = \lim_{(E_1,\geqq)} g_{E_1}$, then x_1 is a cluster point of g; by Theorem 1, x_1 is a cluster point of almost every g_E. Let $A = \{E \mid x_1 \neq \lim_{(E,\geqq)} g_E$ but g_E has a limit point$\}$. If E is in A, let $x = \lim_{(E,\geqq)} g_E$; since there exist disjoint neighborhoods N_1 of x_1 and N of x and since g_E plunges eventually into N, there is an r_1 in E such that $g_E(r) \notin N_1$ if $r > r_1$ and $r \in E$. Hence $g_E^{-1}(N_1)$ is not cofinal in $(E, \geqq)$, so x_1 is not a cluster point of g_E when $E \in A$. Hence $|A| = 0$ by Theorem 1; since $|A_{x_1}| = 0$ also, we see that $|\{E \mid g_E \text{ has a limit}\}| = |A| + |A_{x_1}| = 0$.

Buck notes that the proof of Theorem 1′ can easily be modified to prove another theorem with the same conclusion as that of Theorem 2.

THEOREM 2′. *If $\mathfrak{R}$ has a countable cofinal subset and if X satisfies Hausdorff's second countability condition,*[5] *then g is divergent if and only if almost every g_E is divergent.*

[4] This is the separation condition in a Hausdorff space; however X need not satisfy the other axioms of such a space.

[5] In this system the second countability condition becomes: There exists a countable subset $\{N_i\}$ of subsets of X such that for each x and each neighborhood N of x there is an i such that N_i contains a neighborhood of x and $N_i \subset N$.

LEMMA 4. *If every neighborhood N of x contains a neighborhood N' of x such that for every y in N' there is a neighborhood N_y of y with $N_y \subset N$, then Qg is closed in X.*

If x is in the closure of Qg, then for every neighborhood N of x there is a point y in $N' \cap Qg$; then for every r_0 in R there exists $r_1 \geqq r_0$ such that $g(r_1) \in N_y \subset N$ so $x \in Qg$.

THEOREM 3. *If $\mathcal{R}$ and X satisfy the hypotheses of Theorem 1 and Lemma 4, and if Qg is separable, then $Qg = Qg_E$ for almost every E in $\mathcal{C}$.*

Take a countable dense subset X' of Qg and follow the proof of Theorem 3 of [1], using Theorem 1 and Lemma 4 at the appropriate points. This is much stronger than the corresponding theorem of [1]; the principal extension is that this formulation is valid for all essentially countable index systems rather than for the integers alone. In case $\mathcal{R}$ is the system of integers, this result includes that of [1] since the hypotheses of [1, Theorem 3] imply the hypotheses of Theorem 1 and Lemmas 2 and 4; Theorems 1 and 2 are not generalizations of the corresponding results of [1] but rather are generalizations in a slightly different direction from the case $n=1$ of those theorems.

Note that a metric space satisfies all the hypotheses on X except that on Qg in Theorem 3; there the requirement that X is separable would be a sufficient additional condition. Hence with X metric and $\mathcal{R}$ having a countable cofinal subset, the set Qg used in this paper is equal to the set Pg analogous to Px of [1]. Any countable index system will do for $\mathcal{R}$ as will the system of real numbers ordered by magnitude or the system of n-tuples of real numbers ordered by $(a_1, \cdots, a_n) \geqq (b_1, \cdots, b_n)$ if $a_i \geqq b_i$ for all $i \leqq n$. Another such system is the system of n-tuples $(r_1, \cdots, r_n)$ where $r_i \in \mathcal{R}_i$, an index system with a countable cofinal subset, and where $(r_1, \cdots, r_n) > (r_1', \cdots, r_n')$ if $r_1 > r_1'$ or $r_1 = r_1'$ and $r_2 > r_2'$ or, for some $j \leqq n$, $r_i = r_i'$ for $i < j$ while $r_j > r_j'$. (This is the so-called ordinal or lexicographic product of the systems $\mathcal{R}_i$.) Still another example is the system of pairs of integers where $(i_1, i_2) \geqq (j_1, j_2)$ means that $i_1 = i_2$ and $j_1 \geqq j_2$.

It may be noted by means of Lemmas 1 and 3 that the proofs of Theorems 1 and 2 of [1] also hold when the set $I \times I \times \cdots \times I$ used in [1] as the domain of the function $x = x[i_1, i_2, \cdots, i_n]$ is replaced by $\mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where the $\mathcal{R}_i$ are any index systems with countable cofinal subsystems, providing that $\mathfrak{S}$ is then defined as $\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n$ where $\mathcal{C}_i$ is the family of cofinal subsets of $\mathcal{R}_i$. The theorems thus obtained almost include the corresponding results of both papers, the case $n=1$ giving analogues of the present theorems with stronger hypotheses, the case where all $\mathcal{R}_i = I$ giving those of [1].

An open question is whether the existence of a countable cofinal subset is needed to derive the conclusions of Lemmas 1 and 3; if not, some weakening of the hypotheses of the theorems would be possible.

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