ON THE UNIQUENESS OF YOUNG'S DIFFERENTIALS

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Introduction. The differential of a function of several variables may be defined in a variety of ways, of which the one given by Young[1] renders the best parallelism with the case of a single variable. Stated in the way given below, his definition is applicable to a function defined in a set S of points containing limiting points at which the function is to have differentials. The question of the uniqueness of the differentials, however, arises. In this paper we shall first define, and prove two theorems concerning, the "limiting directions" which describe the directional distribution of the points of S near a limiting point. Then we proceed to show that properties of these limiting directions determine whether the differential is unique or not.

Consider the n-dimensional space. We use the notation for the scalar product of two vectors $a(a_1, a_2, \cdots, a_n)$ and $b(b_1, b_2, \cdots, b_n)$: $a \cdot b = \sum_{i=1}^{n} a_i b_i$. By $\|a\|$ we mean $+(a \cdot a)^{1/2}$; and when $\|a\| = 1$, a is called a direction.

DEFINITION. Let α be a limiting point of S (this will be assumed throughout the paper). If l is a direction such that

$$\left\{ \underset{x \text{ in } S \text{ and } \delta > \|x-\alpha\| > 0}{\text{lower bound of}} \left\| \frac{x-\alpha}{\|x-\alpha\|} - l \right\| \right\} = 0$$

for all positive δ, it is said to be a limiting direction of S at α.

THEOREM 1. *S has at least one limiting direction at α.*

THEOREM 2. *If ω is a direction perpendicular to all of the limiting directions of S at α, then*

$$\lim_{x \text{ in } S,\, x \to \alpha} \omega \cdot \frac{x-\alpha}{\|x-\alpha\|} = 0.$$

Each of the above theorems is readily derived by means of an indirect proof, with the help of the Heine-Borel theorem in the case of Theorem 1.

Let a function $f(x)$ be defined for all x in S. Young's definition of the differential may be stated in the following way:

DEFINITION OF THE DIFFERENTIAL. If there exists a vector V such that

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[1] Proc. London Math. Soc. (2) vol. 7 (1908) p. 157.

$$\lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} = 0,$$

$f(x)$ is said to have a differential $df(\alpha, \eta) = V\cdot\eta$ at α.

THEOREM 3. *The differential $df(\alpha, \eta)$, if it exists, is unique when S has, at α, n linearly independent limiting directions.*

PROOF. Suppose both $U\cdot\eta$ and $V\cdot\eta$ can be written as $df(\alpha, \eta)$. Let $l_1, l_2, \cdots, l_n$ be n linearly independent limiting directions of S at α. The theorem is proved if we can show that $(U-V)\cdot l_i = 0$, $i = 1, 2, \cdots, n$.

To prove this we have

$$\lim_{x \text{ in } S,\, x\to\alpha} (U-V)\cdot\frac{x-\alpha}{\|x-\alpha\|} = \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x)-f(\alpha)-V\cdot(x-\alpha)}{\|x-\alpha\|} - \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x)-f(\alpha)-U\cdot(x-\alpha)}{\|x-\alpha\|} = 0.$$

Hence given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| (U-V)\cdot\frac{x-\alpha}{\|x-\alpha\|} \right| < \epsilon$$

for all x in S satisfying $\delta \geqq \|x-\alpha\| > 0$. Since

$$\left\{ \text{lower bound of}_{\delta > \|x-\alpha\| > 0 \text{ and } x \text{ in } S} \left\| \frac{x-\alpha}{\|x-\alpha\|} - l_i \right\| \right\} = 0,$$

there must be a point x_0 in S such that

$$\delta > \|x_0 - \alpha\| > 0 \quad \text{and} \quad \left\| \frac{x_0-\alpha}{\|x_0-\alpha\|} - l_i \right\| < \epsilon.$$

Thus

$$\begin{aligned} |(U-V)\cdot l_i| &= \left| (U-V)\cdot\left[l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} + \frac{x_0-\alpha}{\|x_0-\alpha\|} \right] \right| \\ &\leqq \left| (U-V)\cdot\left(l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} \right) \right| \\ &\quad + \left| (U-V)\cdot\frac{x_0-\alpha}{\|x_0-\alpha\|} \right| \\ &\leqq \|U-V\| \left\| l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} \right\| + \epsilon \leqq [\|U-V\| + 1]\epsilon. \end{aligned}$$

But ϵ is arbitrary, hence

$$(U - V)\cdot l_i = 0.$$

THEOREM 4. *If S has $m(<n)$ linearly independent limiting directions $l_1, l_2, \cdots, l_m$ at α, and all its other limiting directions are linear combinations of them, and if $V\cdot\eta$ is a differential of $f(x)$ at α, the most general form of the differential is*

$$\left(V + \sum_{i=1}^{n-m} K_i\omega_i\right)\cdot\eta, \tag{A}$$

where K_i are arbitrary constants and $\omega_1, \omega_2, \cdots, \omega_{n-m}$ are linearly independent directions perpendicular to the l's.

PROOF. If $U\cdot\eta$ is also a differential at α, from the proof of the last theorem we have $(U-V)\cdot l_j=0, j=1, 2, \cdots, m$. Thus $(U-V)$ must be a linear combination of the vectors $\omega_1, \omega_2, \cdots, \omega_{n-m}$ which are perpendicular to the l's, that is, U is of the form (A).

To show that $(V+\sum_{i=1}^{n-m}K_i\omega_i)\cdot\eta$ is actually a differential, we have by Theorem 2

$$\begin{aligned}\lim_{x \text{ in } S,\, x\to\alpha} &\frac{f(x) - f(\alpha) - (V + \sum_{i=1}^{n-m}K_i\omega_i)\cdot(x-\alpha)}{\|x-\alpha\|} \\ &= \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} \\ &\quad - \sum_{1}^{n-m} K_i \lim_{x \text{ in } S,\, x\to\alpha} \omega_i\cdot\frac{x-\alpha}{\|x-\alpha\|} = 0.\end{aligned}$$

The theorem follows immediately.

COROLLARY. *There exists a unique differential $U\cdot\eta$ where U is in the m-dimensional space formed by $l_1, l_2, \cdots, l_m$.*

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