THE "FUNDAMENTAL THEOREM OF ALGEBRA" FOR QUATERNIONS

SAMUEL EILENBERG AND IVAN NIVEN

We are concerned with polynomials of degree n of the type

$$f(x) = a_0 x a_1 x \cdots x a_n + \phi(x),$$

where $x, a_0, a_1, \cdots, a_n$ are real quaternions ($a_i \neq 0$ for $i = 0, 1, \cdots, n$) and $\phi(x)$ is a sum of a finite number of similar monomials $b_0 x b_1 x \cdots x b_k$, where $k < n$.

THEOREM 1. *The equation* $f(x) = 0$ *has at least one solution.*[1]

The 4-dimensional euclidean space R_4 of all quaternions will be made compact by adding the point ∞ to form a 4-dimensional sphere S_4. Setting $f(\infty) = \infty$ we get a mapping

$$f: S_4 \to S_4.$$

The continuity of f at ∞ follows from the fact that $|f(x)|$ increases without limit as $|x|$ increases without limit, a fact which is obvious from the definition of f. It should be noted that this argument is not valid for polynomials of degree n with more than one term of degree n.

Theorem 1 is then a consequence of the following theorem which asserts that "essentially" the equation $f(x) = 0$ has exactly n solutions.

THEOREM 2. *The mapping* $f: S_4 \to S_4$ *has the degree* n *(in the sense of Brouwer).*[2]

We define a mapping $g: S_4 \to S_4$ as follows:

$$g(x) = x^n \quad \text{for} \quad x \in R_4, \qquad g(\infty) = \infty.$$

Theorem 2 is a consequence of the following two lemmas.

LEMMA 1. *The mappings* f *and* g *of* S_4 *into* S_4 *are homotopic.*

LEMMA 2. *The mapping* g *has degree* n.

PROOF OF LEMMA 1. Define for $0 \leqq t \leqq 1$

Received by the editors December 27, 1943.

[1] This result has been obtained for the special case in which all terms of $f(x)$ have the form ax^k; cf. Ivan Niven, *Equations in quaternions*, Amer. Math. Monthly vol. 48 (1941) pp. 654–661.

[2] Cf. Alexandroff and Hopf, *Topologie*, Berlin, 1935, chap. 12, for the theory of the degree of a mapping.

$$f_t(x) = a_0 x a_1 x \cdots x a_n + (1 - t)\phi(x) \quad \text{for} \quad x \in R_4,$$
$$f_t(\infty) = \infty.$$

Clearly $f_t(x)$ is continuous in both x and t, and consequently f is homotopic to the mapping

$$f_1(x) = a_0 x a_1 x \cdots x a_n \quad \text{for} \quad x \in R_4,$$
$$f_1(\infty) = \infty.$$

Now for each $i = 0, 1, \cdots, n$ choose a continuous path $a_{i,t}$ in $R_4 - (0)$, where $1 \leqq t \leqq 2$, with end points $a_{i,1} = a_i$ and $a_{i,2} = 1$. Define for $1 \leqq t \leqq 2$

$$f_t(x) = a_{0,t} x a_{1,t} x \cdots x a_{n,t} \quad \text{for} \quad x \in R_4, \ f_t(\infty) = \infty.$$

Again $f_t(x)$ is continuous in x and t and since $f_2(t) = g(t)$ the conclusion follows.

PROOF OF LEMMA 2. It is known[3] that the equation

$$g(x) = i, \quad \text{that is,} \quad x^n = i, \tag{1}$$

has exactly n roots in quaternions, and hence these must be simply the nth roots of i in complex numbers. We shall compute the Jacobian of the mapping function g for the roots of (1), and show that it is positive at each of these roots. This imples our conclusion.[4]

Any quaternion $x = x_1 + x_2 i + x_3 j + x_4 ij$ can be written in the form $x = k(\cos\theta + \tau \sin\theta)$, where

$$k = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$
$$\sin\theta = \pm (x_2^2 + x_3^2 + x_4^2)^{1/2}/k, \qquad \cos\theta = x_1/k,$$
$$\tau = \pm (x_2 i + x_3 j + x_4 ij)/(x_2^2 + x_3^2 + x_4^2)^{1/2}.$$

The last equation is invalid if $x_2^2 + x_3^2 + x_4^2 = 0$; in this case let τ be any unit vector. The sign used in the last two equations is arbitrary; however, in the special case to be considered in which x represents any nth root of i, it will be convenient to let this sign coincide with that of x_2. Now we have

$$x^n = k^n(\cos n\theta + \tau \sin n\theta),$$

and writing $x^n = f_1 + f_2 i + f_3 j + f_4 ij$ we obtain

$$f_1 = f_1(x_1, x_2, x_3, x_4) = k^n \cos n\theta,$$

[3] Cf. Ivan Niven, *The roots of a quaternion*, Amer. Math. Monthly vol. 49 (1942) pp. 386–388, or Louis Brand, a paper with the same title, same journal and volume, pp. 519, 520.

[4] Cf. Alexandroff and Hopf, p. 477.

$$f_r = f_r(x_1, x_2, x_3, x_4) = k^n x_r \sin n\theta / \pm (x_2^2 + x_3^2 + x_4^2)^{1/2} \quad (r = 2, 3, 4).$$

Computing $\partial f_r/\partial x_s$ $(r, s = 1, 2, 3, 4)$, and using the following relations which hold for all the roots of (1), $k=1$, $\sin n\theta = 1$, $\cos n\theta = 0$, $\sin\theta = x_2$, $\cos\theta = x_1$, $x_3 = x_4 = 0$, $\tau = i$, we find that the Jacobian of the system is

$$\begin{vmatrix} nx_2 & -nx_1 & 0 & 0 \\ nx_1 & nx_2 & 0 & 0 \\ 0 & 0 & 1/x_2 & 0 \\ 0 & 0 & 0 & 1/x_2 \end{vmatrix}.$$

Since $x_1^2 + x_2^2 = 1$ and $x_2 \neq 0$, this has the positive value n^2/x_2^2. This completes the proof.

The above proofs can be extended to the case where the coefficients of $f(x)$ belong to the Cayley algebra with 8 units, using mappings of an 8-dimensional sphere S_8 into itself. The following two facts should be kept in mind. (1) Due to the non-associativity of the Cayley numbers each monomial of $f(x)$ must be parenthesized in a definite manner, and a monomial of degree k may contain more than $k+1$ constants. (2) The monomial x^n need not be parenthesized.

The proof also applies equally well to the fundamental theorem of algebra in the usual case of complex variables. The resulting proof will use the degree of a mapping of the 2-sphere S_2 into itself. This proof differs slightly from the topological proof of Alexandroff and Hopf[5] which uses the degree of a mapping of S_1 into itself and the theorem of Rouché (in a topological formulation).

University of Michigan and
Purdue University

[5] Loc. cit. p. 469. Cf. also Wei-Liang Chow, Math. Ann. vol. 116 (1939) p. 463.