TOPOLOGICAL ANALOG OF THE WEIERSTRASS DOUBLE SERIES THEOREM

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Since any light interior transformation of a sphere or a Riemann surface into a sphere is topologically equivalent to an analytic transformation and any non-constant analytic transformation is light and interior,[1] it might be expected in view of the Weierstrass Double Series Theorem that the limit of a uniformly convergent sequence of light interior mappings of a sphere into a sphere would be light and interior. However, this is readily seen not to be the case. For if we let r denote $|z|$ and for each $n>0$ we map the complex sphere onto itself by the function $f_n(z)$ defined by

$$\begin{aligned} w &= z && \text{for } r \geqq 2, \\ w &= z/n && \text{for } r \leqq 1, \\ w &= (r-1)z + (2-r)(z/n) && \text{for } 1 < r < 2, \end{aligned}$$

each mapping is topological, whereas the sequence $f_n(z)$ converges uniformly to the mapping $f(z)$:

$$\begin{aligned} w &= z && \text{for } r \geqq 2, \\ w &= 0 && \text{for } r \leqq 1, \\ w &= (r-1)z && \text{for } 1 < r < 2. \end{aligned}$$

The latter mapping is neither light nor interior. (Of course, analytic mappings topologically equivalent to the transformations $f_n(z)$ can be chosen so as to exhibit a wide range of behaviors, since they need only be topological mappings.)

It will be noted in this example, however, that if we factor the limit mapping $w=f(z)$ into the form $f_2f_1(z)$ where f_1 is monotone and f_2 is light,[2] *the transformation* f_2 *is light and interior*. This suggests the possibility, which will indeed be realized in the much more general situation where the mappings operate on an arbitrary locally connected continuum, that the limit of a uniformly convergent sequence of light interior mappings always factors into a monotone transformation followed by a light interior one. In fact, it will be shown that if the

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[1] See S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris, 1938, Gauthier-Villars.

[2] See, for example, the author's *Analytic topology*, Amer. Math. Soc. Colloquium Publications, New York, 1942, chap. 8. Note references therein also to S. Eilenberg.

individual mappings in the sequence admit of such factorization, so also does their limit.

Now it was shown by Wallace[3] that a continuous transformation $f(A)=B$ on a locally connected continuum A factors into this form if and only if it is quasi-monotone, that is, provided the inverse of each continuum K in B having a non-empty interior relative to B consists of a finite number of components each of which maps onto all of K under f or, equivalently, provided that for each connected open set R in B each component of $f^{-1}(R)$ maps onto R under f. Thus our main conclusion asserts in these terms that the class of quasi-monotone mappings of a locally connected continuum A onto another one B is closed in the space B^A of all mappings of A into B. The Weierstrass Double Series Theorem asserts, of course, that the class of analytic transformations is closed. In view of the equivalence theorem of Stoïlow[1] our result is analogous, except that the "constants" (more precisely, "the continua on which the mappings are constant") must first be factored out.

THEOREM. *The limit mapping $f(A)=B$ of a uniformly convergent sequence of quasi-monotone mappings $f_n(A)=B$ on a locally connected continuum A is itself quasi-monotone.*

In view of the fact mentioned above that a continuous mapping $g(A)=B$ on a locally connected continuum A is quasi-monotone if and only if each component of the inverse of an arbitrary region (that is, connected open subset) R in B maps onto R under g, our theorem is a direct consequence of the following more general one.

THEOREM. *Let the sequence of continuous mappings $f_n(A)=B$ on the locally connected continuum[4] A converge uniformly to the mapping $f(A)=B$. If for each $\epsilon>0$ there exists an integer N such that for any $n>N$ it is true that if R is any region in B and Q is any component of $f_n^{-1}(R)$, then*

$$f_n(Q)\cdot\{R - R\cdot V_\epsilon[F(R)]\} \neq 0$$

implies

$$f_n(Q) \supset R - R\cdot V_\epsilon[F(R)],$$

then f is quasi-monotone.

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[3] A. D. Wallace, *Quasi-monotone transformations*, Duke Math. J. vol. 7 (1940) pp. 136–145.

[4] It is understood that a continuum is compact, metric and connected. For any set X and any $\epsilon>0$, $V_\epsilon(X)$ denotes the set of all points x whose distance $\rho(x, X)$ from X is less than ϵ. Also if G is any open set, $F(G)$ denotes its boundary, that is, $\overline{G}-G=F(G)$.

PROOF. To prove f quasi-monotone it suffices, in view of the factorization process, to show that for any $y \in B$ and any neighborhood V of a component X of $f^{-1}(y)$, $f(V)$ contains a neighborhood of y. By local connectedness, there exists a region U in A such that

$$V \supset \overline{U} \supset U \supset X.$$

Further, since X is a component of $f^{-1}(y)$, we may choose U also so that if $F = F(U)$, $f(F) \cdot y = 0$. Let R be a region in B containing y and such that $\overline{R} \cdot f(F) = 0$. Let

$$\text{(i)} \qquad \begin{aligned} 3\epsilon &= \rho[R, f(F)], \\ F_n &= f_n(F). \end{aligned}$$

By uniform convergence there exists an integer N such that for any $n > N$,

$$\text{(ii)} \qquad \begin{aligned} F_n &\subset V_\epsilon[f(F)], \\ f_n(X) &\subset R. \end{aligned}$$

Now for any $n > N$, let R_n be the component of $B - F_n$ containing R and let Q_n be the component of $f_n^{-1}(R_n)$ containing X. By (ii) we have

$$f_n(Q_n) \cdot R \neq 0.$$

Whence by (i)

$$f_n(Q_n) \cdot \{R_n - R_n \cdot V_\epsilon[F_n]\} \neq 0;$$

and since

$$F_n \supset F(R_n),$$

this gives

$$f_n(Q_n) \cdot \{R_n - R_n \cdot V_\epsilon[F(R_n)]\} \neq 0.$$

By hypothesis this implies that for each $n > N$,

$$f_n(Q_n) \supset R_n - R_n \cdot V_\epsilon[F(R_n)] \supset R.$$

Thus, since $Q_n \subset U$,

$$f_n(\overline{U}) \supset f_n(Q_n) \supset R,$$

for every $n > N$.

Now by uniform convergence, $f_n(\overline{U})$ converges to $f(\overline{U})$. Thus we have

$$f(\overline{U}) = \lim f_n(\overline{U}) \supset R,$$

so that

$$f(V) \supset f(\overline{U}) \supset R,$$

as was to be proven.

COROLLARY. *If the sequence $f_n(A) = B$ of light interior mappings on a locally connected continuum A converges uniformly to the mapping $f(A) = B$, then f admits of unique factorization into the form $f = f_2 f_1$ where f_1 is monotone and f_2 is light and interior.*

The case in which A is a topological sphere is of special interest, not only because of its connection with analytic functions but also because in this case the action of the separate factor mappings can be more completely analyzed. For the monotone factor $f_1(A) = A'$ always yields[5] a cactoid A' and then the light interior factor $f_2(A') = B$ operating on the cactoid A' lends itself readily[6] to analysis.

In conclusion it may be noted that our main conclusion obtained here for monotone mappings is known[7] to hold also for monotone mappings, that is, the limit of a uniformly convergent sequence of monotone mappings of a locally connected continuum A onto another one B is monotone.

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[5] See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 81–88.

[6] See footnote 2, chap. 10.

[7] Footnote 2, p. 174.