periodic under f, then x is also almost periodic under f^n for every integer n.

The proof proceeds easily from Lemma 1, Remark 2, and Theorems 5 and 6.

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SOME PROPERTIES OF SUMMABILITY. II[1]

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1. Summability of bounded sequences. It follows from a well known result of H. Steinhaus[2] that no regular matrix method of summability can be effective for (that is, assign a finite limit to) every element in the space (m) of bounded sequences. The object of this note is to consider some questions suggested by this fact. The first of these may be formulated as follows. If A is a given regular matrix method let J_A denote the set of all A-summable bounded sequences. We then ask what are necessary and sufficient conditions on a subset E of (m) in order that there exist a regular A such that $E \subset J_A$? In Theorem 1 below it is shown that the separability of E is a sufficient condition. It seems unlikely that this condition is necessary although we have been unable to decide the question. It is clearly equivalent to the question of whether every J_A is separable.

THEOREM 1. *Let E be an arbitrary separable subset of (m). Then every regular matrix $A = (a_{mk})$ contains a (necessarily regular) row-submatrix $B = (a_{m_i k})$ such that $E \subset J_B$.*

PROOF. The proof depends on an application of the so-called diagonal process. Let D denote a dense denumerable subset of E of points $x_n \equiv \{s_k^n\}$ for $n = 1, 2, 3, \cdots$, and let $A = (a_{mk})$ be an arbitrary regular matrix. By the Silverman-Toeplitz conditions the sequence $\{t_m^1 \equiv \sum_k a_{mk} s_k^1\}$ is bounded and therefore contains a convergent subsequence $\{t_{m_i}^1\}$, where we may suppose that $m_1 = 1$. Let $a_{m_i k} \equiv a_{ik}^1$ for all i and k, and select a convergent subsequence $\{t_{i_j}^2\}$ from the bounded sequence $\{t_i^2 \equiv \sum_k a_{ik}^1 s_k^2\}$ so that $i_1 = 1$ and $i_2 = 2$. Next let $a_{i_j k}^1 \equiv a_{jk}^2$ for all j and k, proceed to $\{s_k^3\}$, and so forth. We define in this way a sequence of regular matrices $(a_{mk}), (a_{mk}^1), (a_{mk}^2), \cdots$, $(a_{mk}^n), \cdots$, each a row-submatrix of the preceding. The "diagonal matrix" $(a_{mk}^m) \equiv B$ is evidently regular and by its construction it is clearly effective for each x_n in D. To show that B is also effective in $E - D$ we first observe that the operation $y = B(x)$, where $B(x) \equiv \{\sum_k a_{mk}^m s_k\}$ and $x \equiv \{s_k\}$, is linear on (m) to (m). Then if x_0 is any point of $E - D$ there exists a subsequence $\{x_{n_i}\}$ of D that converges to x_0. Setting $y_i \equiv B(x_{n_i})$ we have $y_0 \equiv \lim_i y_i = \lim_i B(x_{n_i}) = B(x_0)$. Since y_i belongs to (c), a closed subset of (m), it follows that y_0 belongs to (c). This completes the proof.

2. **A property of reversible methods.** Let us first recall the following definitions. The method A corresponding to the matrix (a_{mk}) is said to be *reversible*[3] if the system of equations

$$(2.1) \qquad\qquad t_m = \sum_{k=1}^{\infty} a_{mk} s_k \qquad\qquad (m = 1, 2, 3, \cdots)$$

has a unique solution $\{s_k\}$ corresponding to each convergent sequence $\{t_m\}$. The method A is said to be of[3] *type M* if the conditions

$$\sum_{m=1}^{\infty} |u_m| < \infty, \qquad \sum_{m=1}^{\infty} u_m a_{mk} = 0 \qquad\qquad (k = 1, 2, 3, \cdots)$$

always imply $u_m = 0$ $(m = 1, 2, 3, \cdots)$. Finally, every method A which is simultaneously regular, reversible, and of type M, is said to be *perfect*[3].

We now ask if every regular method of summability which is stronger than convergence is necessarily effective for some *unbounded* sequences. For the class of reversible matrix methods an affirmative answer is supplied by the following theorems. These theorems bring to light the fact that while perfect methods of summability may be equivalent to convergence, reversible regular methods not of type M are always stronger than convergence.

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[3] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 90.

THEOREM 2. *Every perfect method of summability which is stronger than convergence is effective for some unbounded sequences.*

PROOF. Let $A = (a_{mk})$ be an arbitrary perfect matrix, stronger than convergence, and let F_A denote the set of all A-summable sequences. Then the transformation (2.1) of F_A into (c) has an inverse of the form[4]

$$(2.2) \qquad\qquad s_k = \sum_{m=0}^{\infty} A_{km} t_m \qquad\qquad (k = 1, 2, 3, \cdots),$$

where $t_0 \equiv \lim_m t_m$ and $\sum_{m=0}^{\infty} |A_{km}| < \infty$ $(k = 1, 2, 3, \cdots)$. If we let $x \equiv \{s_k\}$ and $y \equiv \{t_m\}$ we may express (2.1) as $y = A(x)$ and (2.2) as $x = A^{-1}(y)$.

Since A is of type M it follows from a lemma of Banach[5] that as x ranges over (c) the set G_A of all points $y = A(x)$ is dense in (c).

To give an indirect proof we assume now that $F_A \subset (m)$, that is, that every A-summable sequence is bounded. As a consequence it follows easily that the operation $x = A^{-1}(y)$ is then linear on (c) to F_A. Let x_0 be an arbitrary point of F_A and set $y_0 = A(x_0)$. Since y_0 is in (c) there exists, by the foregoing remark, a sequence $\{y_n\}$ in G_A, where $y_n = A(x_n)$ for certain x_n in (c), such that $y_n \to y_0$. But then $x_n = A^{-1}(y_n) \to A^{-1}(y_0) = x_0$, and since (c) is complete we see that x_0 belongs to (c). Thus A is equivalent to convergence, and this contradiction completes the proof.

THEOREM 3. *Every reversible regular method A which is not of type M is effective for some unbounded sequences.*

PROOF. As x ranges over (m) we denote by H_A the set of all points $y = A(x)$. By an easy extension of the previously mentioned lemma of Banach, it has been shown[6] that A is of type M if and only if the points $z_n \equiv \{\delta_k^n\}$ for $n = 1, 2, 3, \cdots$ (where δ_k^n is the Kronecker symbol) are points of $\bar{H}_A$, the closure of H_A. Hence if A is not of type M at least one of the points z_n, say z_r, is not in $\bar{H}_A$. Since A is reversible and z_r belongs to (c), the equation $A(x) = z_r$ has a unique solution $x = x_r$. It is clear that x_r can not be a point of (m) and this fact establishes the theorem.

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[4] See J. D. Hill, *Some properties of summability*, Duke Math. J. vol. 9 (1942) pp. 373–381; in particular, p. 376.

[5] See footnote 3, p. 93, Lemma 2.

[6] See J. D. Hill, *On perfect methods of summability*, Duke Math. J. vol. 3 (1937) pp. 702–714; in particular p. 704, Theorem 4.

An alternative proof of this theorem may be given as follows. Under the assumed hypotheses on A we conclude from known theorems that[7] there exists a regular method B, not weaker than A, with which A is inconsistent; and that[8] A and B are consistent for bounded sequences. Hence the inconsistency must occur for unbounded sequences.

3. **Methods effective for no bounded divergent sequences.** As our final question we ask if there exist regular methods, stronger than convergence, which are effective for *no* bounded divergent sequences. An example will suffice to show that the answer is affirmative.

Example. We define matrices $B = (b_{mk})$ and $C = (c_{mk})$ as follows: $b_{11} \equiv b_{k+1,k} \equiv 1$ $(k = 1, 2, 3, \cdots)$, and $b_{mk} \equiv 0$ otherwise; $c_{mm} \equiv 1/m$ $(m = 1, 2, 3, \cdots)$, and $c_{mk} \equiv 0$ otherwise. The matrix B is a regular matrix that leaves invariant the limit points of each bounded sequence. The matrix C, on the other hand, assigns the limit 0 to every bounded sequence. Consequently, the sum matrix $A \equiv B + C$ leaves invariant the limit points of each bounded sequence, and is therefore effective for no bounded divergent sequence. One easily verifies, however, that A assigns the limit 0 to the unbounded sequence $\{(-1)^{k-1}k!\}$.

The matrix A of this example is a simple instance of a Raff matrix,[9] namely, a matrix that leaves invariant the limit points of each bounded sequence. Every Raff matrix which is stronger than convergence will therefore possess the property in question, for example, every reversible Raff matrix which is not of type M (see Theorem 3 above).

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[7] See footnote 4, p. 380, Theorem 5.
[8] See footnote 3, p. 95, Théorème 11.
[9] Hermann Raff, *Zur Theorie der lineare Transformationen*, Math. Zeit. vol. 37 (1933) pp. 572–577.