$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p + 2\nu + 1}$$

if E has the form II; *and*

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{n} \left(\frac{\alpha_k \beta_\nu}{2p + 2\nu + 1}\right) u_{p+k+\nu+1}$$
$$+ \gamma_0 u_p + \sum_{k=1}^{m+n} \left(\gamma_k - \frac{2}{re^{i\theta}} \gamma_{k-1}\right) u_{p+k}$$

if E has the form III.

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ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

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The well known Gauss' Quadrature Formula

$$\int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^{n} \rho_i^{(n)} G_k(\xi_i^{(n)}) \tag{1}$$

is valid for every polynomial $G_k(x)$, of degree $k \leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being the roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ $(i=1, 2, \cdots, n; n=1, 2, \cdots)$.[1] If the sequence $\{P_n(x)\}$ is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers $\rho_i^{(n)}$, $i=1, 2, \cdots, n$, are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$\int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \qquad k \leqq 2n - 1; n = 1, 2, \cdots. \tag{2}$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).[2]

We shall give here four distinct proofs of this statement, without imposing any restrictions on $\psi(x)$.

Received by the editors June 1, 1943.

[1] $\psi(x)$ is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: $c_n = \int_{-\infty}^{\infty} x^n d\psi(x)$; $n=0, 1, 2, \cdots$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

Consider the sequence of complex numbers $\{c_n\}$, $n=0, 1, 2, \cdots$, subject to the conditions

$$(3) \qquad \Delta_{n+1} \equiv \left| c_{i+k} \right|_{i,k=0}^{n} \neq 0; \qquad n = 0, 1, 2, \cdots .$$

Consider also the Stieltjes linear functional σ, with

$$(4) \qquad \sigma(x^k) = c_k; \qquad k = 0, 1, 2, \cdots ,$$

and the polynomials $\{P_n(x)\}$, $n=0, 1, 2, \cdots$, orthogonal relative to the sequence $\{c_n\}$, that is [3, 4]

$$(5) \qquad \sigma\{P_n(x)P_m(x)\} = \begin{cases} 0, & m \neq n, \\ h_n \equiv \Delta_{n+1}/\Delta_n \neq 0, & m = n. \end{cases}$$

Let $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, be the roots, distinct or not, of $P_n(x)$, $n=1, 2, \cdots$.

I. The first method of proving our statement consists in proving the following theorem.

THEOREM. *From the validity of the formula*

$$(6) \qquad \rho_n \sum_{i=1}^{n} \left(\xi_i^{(n)}\right)^k = c_k, \qquad n \geqq \left[\frac{k+1}{2}\right],$$

for $k \leqq 2$, it follows that the $\{P_n(x)\}$ are the Tchebycheff polynomials, so that (6) *holds for all integral k.*

Introduce the mean of order ν of the numbers $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$:

$$(7) \qquad \mu_\nu^{(n)} = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\xi_i^{(n)}\right)^\nu\right]^{1/\nu}, \qquad \nu = 1, 2, \cdots ; n = 1, 2, \cdots .$$

On equating the coefficients of x^{n-1} and x^{n-2} on both sides of the recurrence relation

$$(8) \qquad \begin{gathered} P_n(x) = (x - \alpha_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \\ \lambda_n \neq 0, \quad P_{-1} \equiv 0, n = 1, 2, \cdots , \end{gathered}$$

we get

$$(9) \qquad \delta_1^{(n)} = -\sum_{k=1}^{n} \alpha_k, \qquad \delta_2^{(n)} = -\sum_{k=2}^{n} \lambda_k + \sum_{r=2}^{n} \alpha_r \sum_{k=1}^{r-1} \alpha_k,$$

where we let

$$(9') \qquad P_n(x) = x^n + \delta_1^{(n)} x^{n-1} + \delta_2^{(n)} x^{n-2} + \cdots + \delta_n^{(n)}.$$

On putting $k=0, 1, 2$ in (6) we obtain

$$\rho_n = c_0/n; \quad \mu_1^{(n)} = c_1/c_0, \; n \geqq 1; \quad \mu_2^{(n)} = c_2/c_0, \; n \geqq 2. \tag{10}$$

Since, by (9) and (9′),

$$\begin{aligned} n\mu_1^{(n)} &= -\delta_1^{(n)} = \sum_{k=1}^{n} \alpha_k, \\ n(\mu_2^{(n)})^2 &= (\delta_1^{(n)})^2 - 2\delta_2^{(n)} = \left(\sum_{k=1}^{n} \alpha_k\right)^2 + 2\sum_{k=1}^{n} \lambda_k - 2\sum_{r=2}^{n}\sum_{k=1}^{r-1} \alpha_k, \end{aligned} \tag{11}$$

we find:

$$\alpha_1 = (\alpha_1 + \alpha_2)/2 = (\alpha_1 + \alpha_2 + \alpha_3)/3 = \cdots,$$

whence we have the fundamental result:

$$\alpha_1 = \alpha_2 = \cdots = \alpha, \tag{12}$$

and similarly,

$$\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda. \tag{12′}$$

Here α and λ are independent of n. The solution of (8) under these conditions is

$$\begin{aligned} P_n(x) = {} & [(x - \alpha + ((x-\alpha)^2 - 4\lambda)^{1/2})^n \\ & + (x - \alpha - ((x-\alpha)^2 - 4\lambda)^{1/2})^n]/2^n, \quad n = 1, 2, \cdots, \end{aligned} \tag{13}$$

which shows that the $\{P_n(x)\}$ are the Tchebycheff polynomials, whence the validity of (6) for $k=3, 4, \cdots$ follows.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the arithmetic mean and the root-mean-square of the roots* $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, *of the orthogonal polynomials* $P_n(x)$, $n=1, 2, \cdots$, *do not depend on* n, *the means of all orders* $\nu \leqq 2n-1$ *possess the same property, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

II. Define

$$R_{n-1}(y) = \sigma\{(P_n(y) - P_n(x))/(y - x)\},$$

a polynomial of degree $n-1$; $n=1, 2, \cdots$. Our second proof consists in showing that (6) implies

$$R_{n-1}(y) \equiv (c_0/n)P_n'(y), \qquad n \geqq 2. \tag{14}$$

From (6), wherein ρ_n necessarily equals c_0/n, we get, by virtue of (4),

$$\sigma(x^k) = \frac{c_0}{n}\sum_{i=1}^{n}(\xi_i^{(n)})^k; \quad \sigma(G_k(x)) = \frac{c_0}{n}\sum_{i=1}^{n}G_k(\xi_i^{(n)}), \qquad n \geqq \left[\frac{k+1}{2}\right].$$

The desired relation

$$R_{n-1}(y) = \frac{c_0}{n}\sum_{i=1}^{n}\frac{P_n(y) - P_n(\xi_i^{(n)})}{y - \xi_i^{(n)}} = \frac{c_0}{n}P_n(y)\sum_{i=1}^{n}\frac{1}{y - \xi_i^{(n)}} \equiv \frac{c_0}{n}P_n'(y)$$

follows. Now there are different methods of showing that (14) implies (13).

II_1. This statement is proved in a note [2] as a particular case of more general theorems.

II_2. Introduce the polynomials $\{P_n^{(i)}(x)\}$, $i=1, 2, \cdots, n$—the denominators of the convergents of order n of the continued fractions[3]

$$(15) \qquad \frac{\lambda_i|}{|x - \alpha_i} - \frac{\lambda_{i+1}|}{|x - \alpha_{i+1}} - \cdots, \qquad i = 1, 2, \cdots.$$

We have $P_n^{(1)}(x) \equiv P_n(x)$; $P_n^{(2)}(x) \equiv (1/c_0)R_n(x)$, $n=0, 1, 2, \cdots$.

It is easy to show that

$$(16) \qquad P_n(x) = (x - \alpha_1)P_{n-1}^{(2)}(x) - \lambda_2 P_{n-2}^{(3)}(x), \qquad n = 2, 3, \cdots.$$

On the other hand

$$(17) \qquad R_{n-1}(x) = (x - \alpha_n)R_{n-2}(x) - \lambda_n R_{n-3}(x), \qquad n = 3, 4, \cdots.$$

Using (8), (14) and (17), we find that

$$(18) \qquad P_{n-1}(x) = (x - \alpha_n)P_{n-2}^{(2)}(x) - 2\lambda_n P_{n-3}^{(2)}(x), \qquad n = 3, 4, \cdots,$$

which gives, in conjunction with (16),

$$(19) \qquad (\alpha_1 - \alpha_n)P_{n-2}^{(2)}(x) = 2\lambda_n P_{n-3}^{(2)}(x) - \lambda_2 P_{n-3}^{(3)}(x), \qquad n = 3, 4, \cdots.$$

Hence

$$(20) \qquad \begin{gathered} \alpha_3 = \alpha_4 = \cdots = \alpha_1 = \alpha; \qquad \lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda; \\ P_{n-3}^{(2)}(x) \equiv P_{n-3}^{(3)}(x). \end{gathered}$$

This identity, for $n=4$, gives $\alpha_2 = \alpha$, and thus we have arrived again at (12, 12′).

II_3. On putting

[3] They were introduced by Stieltjes [5]; cf. also Perron [6].

$$R_n(y) = c_0(y^n + d_1^{(n)} y^{n-1} + \cdots + d_n^{(n)}), \tag{21}$$

we find from (17) the relations

$$d_1^{(n-1)} = -\sum_{k=2}^{n} \alpha_k, \qquad d_2^{(n-1)} = \sum_{k=3}^{n} \alpha_k \sum_{r=2}^{k-1} \alpha_r - \sum_{k=3}^{n} \lambda_k \tag{22}$$

analogous to (9). From (14) we find

$$nd_1^{(n-1)} = (n-1)\delta_1^{(n)}, \tag{23}$$

whence again $\alpha_1 = \alpha_2 = \cdots = \alpha$; and in the same way, the condition

$$nd_2^{(n-1)} = (n-2)\delta_2^{(n)} \tag{24}$$

again implies $\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda$.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the three highest coefficients of the polynomials* $\{nR_{n-1}(x)\}$ *and* $\{c_0 P_n'(x)\}$, $n = 1, 2, \cdots$, *coincide, then these polynomials are identical, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

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