A CONVERGENCE THEOREM FOR LEBESGUE-STIELTJES INTEGRALS

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This paper deals with the following theorem.

THEOREM. *Let $g(x)$ be a bounded Borel measurable function defined everywhere on $(-\infty, \infty)$. Let $p_n(x)$ be a sequence of normalized functions,*[1] *$p_n \in V$, such that*

$$\sum_{n=1}^{\infty} V(p_n) < \infty. \tag{1}$$

Then $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent to a normalized function $p(x) \in V$, and

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{2}$$

We shall first prove this theorem in the following special cases:

(a) $g(x)$ is a bounded piecewise absolutely continuous function[2] in $(-\infty, \infty)$.

(b) $g(x)$ is continuous in a finite interval and vanishes identically outside this finite interval. (It need not necessarily be continuous at the end points of the interval.)

(c) $g(x)$ is a bounded continuous function in $(-\infty, \infty)$.

First let us prove our assertion concerning $p(x)$. Since $p_n(x)$ are normalized, we have[3] $\sum_{n=1}^{\infty} |p_n(x)| \ll \sum_{n=1}^{\infty} P_n \ll \sum_{n=1}^{\infty} V(p_n)$ where P_n is the upper bound of $|p_n(x)|$ for $-\infty < x < \infty$. Because of (1), it follows that the series $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent. Let $p(x)$ be the limit function. Evidently $p(x)$ is right-continuous and normalized. To show that $p(x) \in V$ it is sufficient to show that $p(x)$ is of bounded variation on $(-\infty, \infty)$. For $\xi > 0$ and any subdivision of $(-\xi, \xi)$, $-\xi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \xi$, we

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[1] $p(x) \in V$ means that $p(x)$ is right-continuous and of bounded variation on the infinite interval $(-\infty, \infty)$. It is normalized if $p(0) = 0$. $V(p_n)$ denotes the total variation of p_n over $(-\infty, \infty)$.

[2] $f(x)$ is piecewise absolutely continuous in $(-\infty, \infty)$ if we can divide $(-\infty, \infty)$ into a finite number of intervals such that in each of these intervals $f(x)$ is absolutely continuous.

[3] "$<<$" is to be read "is dominated termwise by."

have

$$\sum_{i=0}^{m} | p(x_{i+1}) - p(x_i) | \leqq \sum_{n=0}^{\infty} \sum_{i=0}^{m} | p_n(x_{i+1}) - p_n(x_i) |.$$

This implies that[4] $V(p; -\xi, \xi) \leqq \sum_{n=1}^{\infty} V(p_n; -\xi, \xi) \leqq \sum_{n=1}^{\infty} V(p_n)$. Hence $V(p) < \infty$.

Let x_i ($i = 1, 2, \cdots, N$) be the points of discontinuity of $g(x)$ under condition (a), where $x_1 < x_2 < \cdots < x_N$. Then $g(x)$ is absolutely continuous on $x_i < x < x_{i+1}$ and $g(x) = \int_{x_i+0}^{x} g_1'(x)dx + g(x_i+0)$, where $g_1(x)$ is the absolutely continuous part of $g(x)$ in the Lebesgue decomposition. For the infinite interval we then have

$$\int_{-\infty}^{\infty} dg(x) = \int_{-\infty}^{\infty} g_1'(x)dx + \sum_{i=1}^{N} j(x_i)$$

where $j(x_i)$ is the jump of $g(x)$ at $x = x_i$. Since $g(x)$ is bounded, it is evident that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists. In addition we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} | g(x) | | dp_n(x) | \leqq G \cdot \sum_{n=1}^{\infty} V(p_n) < \infty \tag{3}$$

where $G = \text{u.b.}_{-\infty < x < \infty} | g(x) |$. Hence $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x)$ exists.

We shall now show that (2) holds for case (a). Let A be any positive number. Then because of the above remark we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= \sum_{n=1}^{\infty} \Big\{ g(A)p_n(A) - g(-A)p_n(-A) \\ &\qquad - \int_{-A}^{A} p_n(x)dg(x) \\ &= g(A)p(A) - g(-A)p(-A) \\ &\qquad - \sum_{n=1}^{\infty} \int_{-A}^{A} p_n(x)g_1'(x)dx - \sum_{n=1}^{\infty} \sum_{i=1}^{N} p_n(x_i)j(x_i). \end{aligned} \tag{4}$$

By the general convergence theorem[5] of Lebesgue, this becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= g(A)p(A) - g(-A)p(-A) \\ &\qquad - \int_{-A}^{A} p(x)g_1'(x)dx - \sum_{i=1}^{N} p(x_i)j(x_i) \\ &= \int_{-A}^{A} g(x)dp(x). \end{aligned} \tag{5}$$

[4] $V(p_n; -\xi, \xi)$ denotes the total variation of p_n over $(-\xi, \xi)$.

[5] E. C. Titchmarsh, *Theory of functions*, Oxford, 2nd ed., p. 345.

Letting $A \to \infty$, we get

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{6}$$

Now consider

$$\begin{aligned}\sum_{n=1}^{\infty} \left| \int_{-A}^{A} g(x)dp_n(x) \right| &\ll \sum_{n=1}^{\infty} \int_{-A}^{A} |g(x)|\,|dp_n(x)| \\ &\ll G\cdot \sum_{n=1}^{\infty} \int_{-A}^{A} |dp_n(x)| \\ &\ll G\cdot \sum_{n=1}^{\infty} V(p_n).\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x)$ is absolutely and uniformly convergent in A. Hence by a known theorem,[6] we have

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x).$$

Combining this with (6), we get

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x),$$

which concludes case (a).

To prove the theorem for case (b), it is sufficient to show that the expression

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right|$$

can be made arbitrarily small for all sufficiently large values of N. Let $[M_1, M_2]$ be the interval within which $g(x)$ is continuous. By the Weierstrass approximation theorem, there exists a sequence of polynomials $g_m(x)$ which converge uniformly to $g(x)$ in $[M_1, M_2]$. Consider the functions defined by

$$f_m(x) = \begin{cases} g_m(x), & M_1 \leqq x \leqq M_2, \\ 0, & \text{otherwise}. \end{cases}$$

We note that $f_m(x)$ for $m=1, 2, \cdots$ all satisfy the hypothesis (a). Hence we have

[6] See Hobson, *Theory of functions of a real variable*, vol. 2, p. 121.

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_m(x)dp_n(x) = \int_{-\infty}^{\infty} f_m(x)dp(x) \tag{7}$$

for $m=1, 2, \cdots$. Furthermore we have

$$\begin{aligned}
&\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| \\
&\qquad \leqq \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| \\
&\qquad\quad + \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x) - \int_{-\infty}^{\infty} f_m(x)dp(x)\right| \\
&\qquad\quad + \left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right|.
\end{aligned} \tag{8}$$

Because of (7) it follows that

$$\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| < \epsilon_N(m), \tag{9}$$

where $\epsilon_N(m)\rightarrow 0$ as $N\rightarrow\infty$ for every $m=1, 2, \cdots$. Since $f_m(x)$ converges uniformly to $g(x)$ in $[M_1, M_2]$ we also have

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| \\
&\qquad = \left|\int_{M_1}^{M_2} \{g_m(x) - g(x)\}dp(x)\right| \leqq \epsilon_m\cdot V(p) = \epsilon_m',
\end{aligned} \tag{10}$$

where $\epsilon_m = \max_{M_1\leqq x\leqq M_2}|g_m - g|$ and where $\epsilon_m'\rightarrow 0$ when $m\rightarrow\infty$ since $V(p)$ is finite. Also

$$\begin{aligned}
&\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| \\
&\qquad \leqq \left|\sum_{n=1}^{N}\int_{M_1}^{M_2} \{g(x) - g_m(x)\}dp_n(x)\right. \\
&\qquad \leqq \epsilon_m\cdot\sum_{n=1}^{N}\int_{M_1}^{M_2} \left|dp_n(x)\right| \\
&\qquad \leqq \epsilon_m\cdot\sum_{n=1}^{\infty} V(p_n) = \epsilon_m''
\end{aligned} \tag{11}$$

where $\epsilon_m''\rightarrow 0$ when $m\rightarrow\infty$, since $\sum_{n=1}^{\infty} V(p_n)$ is finite.

Now let δ be an arbitrary positive number and take m large enough so that

$$\epsilon_m' + \epsilon_m'' < 2\delta/3. \tag{12}$$

With m fixed, take N large enough so that

$$\epsilon_{N+\nu}(m) < \delta/3, \qquad \nu = 0, 1, 2, \cdots. \tag{13}$$

It follows that [using (12) and (13) in (8)]

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right| < \delta$$

for all sufficiently large N and arbitrary $\delta > 0$.

Coming to case (c), let α be a positive number and define

$$g_\alpha(x) = \begin{cases} g(x), & -\alpha \leqq x \leqq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g_\alpha(x)$ satisfies the conditions of hypothesis (b), we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) = \int_{-\infty}^{\infty} g_\alpha(x)dp(x). \tag{14}$$

By dominated convergence we have

$$\lim_{\alpha=\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{15}$$

Let $G = \text{u.b.}_{-\infty<x<\infty} |g(x)|$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \right| &\ll \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} | g(x) | \, | dp_n(x) | \\ &\ll G \cdot \sum_{n=1}^{\infty} V(p_n). \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x)$ converges absolutely and uniformly in α, and it follows that

$$\begin{aligned} \lim_{\alpha=\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) &= \sum_{n=1}^{\infty} \lim_{\alpha=\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x), \end{aligned} \tag{16}$$

the last step being justified by the theorem on dominated convergence. Combining (14), (15) and (16), we obtain (2). This proves the theorem under hypothesis (c).

To complete the proof of the theorem we need the following lemma.

LEMMA. *Let $p_n(x)$ satisfy the conditions of the theorem. Suppose* (2) *is true for all bounded functions $g(x)$ of Baire class less than α. Then* (2) *is true for all bounded functions $g(x)$ in the class α.*

Let $f_j(x)$ be any convergent sequence in a Baire class $\alpha_1 < \alpha$ such that $f_j(x)$ is uniformly bounded. Let $\lim_{j=\infty} f_j(x) = f^*(x)$. Clearly $f^*(x)$ is also bounded. By hypothesis, we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_j(x) dp_n(x) = \int_{-\infty}^{\infty} f_j(x) dp(x) \tag{17}$$

for $j = 1, 2, \cdots$. By a theorem on bounded convergence, we have

$$\lim_{j=\infty} \int_{-\infty}^{\infty} f_j(x) dp(x) = \int_{-\infty}^{\infty} f^*(x) dp(x). \tag{18}$$

Consider also

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_{-\infty}^{\infty} f_j(x) dp_n(x) \right| &\ll F \cdot \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} | dp_n(x) | \\ &= F \cdot \sum_{n=1}^{\infty} V(p_n), \end{aligned}$$

where F is $\text{u.b.}_{-\infty < x < \infty,\, j=1,2,\ldots} |f_j(x)|$. Hence $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_j(x) dp_n(x)$ converges absolutely and uniformly in j. Therefore

$$\begin{aligned} \lim_{j=\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_j(x) dp_n(x) &= \sum_{n=1}^{\infty} \lim_{j=\infty} \int_{-\infty}^{\infty} f_j(x) dp_n(x) \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f^*(x) dp_n(x). \end{aligned} \tag{19}$$

Combining equations (17), (18) and (19), we get

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f^*(x) dp_n(x) = \int_{-\infty}^{\infty} f^*(x) dp(x),$$

which completes the proof of the lemma.

Finally we return to the original hypothesis. It is known[7] that the

[7] For example, see de la Vallée Poussin, *Intégrales de Lebesgue* (Borel Monograph), 1916, pp. 36, 37.

set of bounded functions in classes 0, 1, 2, $\cdots$ of Baire is the same as the set of bounded Borel measurable functions. Under (c) we have proved the theorem for any bounded continuous function, that is, for bounded functions in class 0 of Baire. By the lemma it holds then for functions in classes 0, 1, 2, $\cdots$ of Baire and hence for all bounded Borel measurable functions. This completes the proof of the theorem.

We note in the theorem above that the boundedness of $g(x)$ was sufficient to insure the existence of the integral $\int_{-\infty}^{\infty} g(x)dp(x)$, provided that $p(x)$ satisfied the conditions of this theorem. Now we shall prove a partial converse of this.

THEOREM. *Let $g(x)$ be a given Borel measurable function defined everywhere in $(-\infty, \infty)$, with the property that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists*[8] *whenever $p(x) \in V$. Then $g(x)$ is bounded everywhere in $(-\infty, \infty)$.*

Let E_n be the set of points x for which $n \leqq g(x) < n+1$. We shall show that the family of non-empty sets E_{n_i} is finite.

Suppose the contrary. Then there must exist an infinity of distinct x_{n_i} since E_{n_i} are mutually disjoint. Consider a particular sequence $x_{n_0}, x_{n_1}, x_{n_2}, \cdots$, one x_{n_i} from each non-empty E_{n_i}.

At this point let us note that given any sequence of integers $0 < n_1 < n_2 < \cdots$ there exists a number ξ such that $\sum_{i=1}^{\infty} 1/n_i^{\xi}$ diverges and $\sum_{i=1}^{\infty} 1/n_i^{\xi+1}$ converges. In class L put all real numbers p such that $\sum_{i=1}^{\infty} 1/n_i^{p}$ is divergent. Put all others in R-class. To the R-class belongs the number 2. The number 0 belongs to the L-class. If p belongs to L, then all numbers less than p also belong to L. Thus we have a Dedekind cut. Let c be the number defined by this cut. Then $\sum_{i=1}^{\infty} 1/n_i^{c-1/2}$ diverges, while $\sum_{i=1}^{\infty} 1/n_i^{c+1/2}$ converges. Hence $\xi = c - 1/2$ satisfies the required condition.

With this in mind, let us return to the proof of the theorem. Using the sequence x_{n_i} defined previously let us form the function

$$p(x) = \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} \omega_{n_i}(x),$$

where

$$\omega_{n_i}(x) = \begin{cases} 0, & x < x_{n_i}, \\ 1, & x \geqq x_{n_i}, \end{cases}$$

and ξ is such a number that $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi+1}$ converges and $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi}$ diverges.

Since $\omega_{n_i}(x)$ is right-continuous and the series uniformly conver-

[8] M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, vol. 15, 1932, p. 206.

gent, $p(x)$ is also right-continuous. Also $p(x)$ is a monotone increasing function. To show that $p(x)\in V$, we make use of the theorem proved earlier. In fact we have

$$\begin{aligned}\int_{-\infty}^{\infty}|dp(x)| = \int_{-\infty}^{\infty}dp(x) &= \int_{-\infty}^{\infty}d\left\{\sum_{i=0}^{\infty}\frac{1}{(n_i+1)^{\xi+1}}\omega_{n_i}(x)\right\}\\ &= \sum_{i=0}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(n_i+1)^{\xi+1}}d\omega_{n_i}(x)\\ &= \sum_{i=0}^{\infty}\frac{1}{(n_i+1)^{\xi+1}}.\end{aligned}$$

To complete the proof of this theorem it is sufficient to show that $\int_{-\infty}^{\infty}g(x)dp(x)$ does not exist for this $p(x)$. Let

$$g_M(x) = \begin{cases} g(x), & \text{if } |g(x)| \leqq M, \\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty}|g_M(x)||dp(x)| &= \sum_{i=0}^{\infty}\frac{1}{(n_i+1)^{\xi+1}}\int_{-\infty}^{\infty}|g_M(x)|\,d\omega_{n_i}(x)\\ &= \sum_{i=0}^{\infty}\frac{1}{(n_i+1)^{\xi+1}}|g_M(x_{n_i})|\\ &\geqq \sum_{n_i<M}\frac{n_i}{(n_i+1)^{\xi+1}},\end{aligned}$$

the first step being justified by the first theorem. Hence

$$\lim_{M=\infty}\sum_{n_i<M}\frac{n_i}{(n_i+1)^{\xi+1}} \leqq \int_{-\infty}^{\infty}|g(x)||dp(x)|,$$

which proves that the integral does not exist, since the series $\sum_{i=0}^{\infty}1/(n_i+1)^{\xi}$ was constructed to be divergent.

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