ON CERTAIN PAIRS OF SURFACES IN ORDINARY SPACE

BUCHIN SU

1. **Introduction.** In a recent paper[1] Jesse Douglas has proposed and solved the following problem: To determine the form of the linear element of a surface in ordinary space upon which exists a family of ∞^2 curves possessing two properties: (1) The angular excess of any triangle ABC formed by curves of the family $\mathcal{J}$ is proportional to the area of the triangle:

$$(1)\qquad \mathcal{E} = A + B + C - \pi = k\mathcal{A},$$

where k denotes a constant; (2) The curves of $\mathcal{J}$ are a *linear* system; that is, a point transformation exists which converts them into the straight lines of a plane. It is natural to inquire what class of surfaces we shall obtain if, instead of using property (2), we make the less specific demand that a point transformation exists which converts the curves of $\mathcal{J}$ into the *geodesics* of another surface. Here we have found certain pairs of surfaces S and S_1 which furnish the complete solution of our generalized problem. According to whether the constant k is zero or not, the linear elements of S and S_1 take different types, whose derivation constitutes the purpose of the present paper.

2. **Conditions for the property $\mathcal{E} = k\mathcal{A}$.** As was shown by Douglas,[2] the necessary and sufficient conditions that every curve of a family $\mathcal{J}$ upon a surface S should have the property $\mathcal{E} = k\mathcal{A}$ can be expressed by the relation

$$(2)\qquad ds/\rho = Pdu + Qdv,$$

where $1/\rho$ is the geodesic curvature of the curve and P, Q obey the condition.

$$(3)\qquad Q_u - P_v = (k - K)W.$$

For the subsequent discussion it is convenient to consider both surfaces S and S_1, wherein the curves of $\mathcal{J}$ upon S correspond to the geodesics of S_1. Let (u, v) be general coordinates of the corresponding points on these surfaces, so that the first fundamental form of S is

Received by the editors February 1, 1943.

[1] J. Douglas, *A new special form of the linear element of a surface*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 101–116.

[2] Douglas, loc. cit., p. 108.

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{4}$$

and that of S_1 is

$$ds_1^2 = E_1du^2 + 2F_1dudv + G_1dv^2. \tag{5}$$

According to the classical theorem of Tissot,[3] there exists upon S one and, in general, only one orthogonal system of curves which corresponds to an orthogonal system upon S_1. Suppose that these surfaces S and S_1 are referred to the orthogonal curves which correspond to each other; then we have

$$ds^2 = Edu^2 + Gdv^2, \tag{6}$$

$$ds_1^2 = E_1du^2 + G_1dv^2. \tag{7}$$

In orthogonal coordinates (u, v), the geodesic curvature $1/\rho$ of any curve $v=v(u)$ upon the surface S is given by[4]

$$\begin{aligned} ds/\rho = {} & (EG)^{-1/2}(E + Gv'^2)^{-1}\{EGv'' - (1/2)EE_v \\ & + (EG_u - (1/2)GE_u)v' + ((1/2)EG_v - GE_v)v'^2 + (1/2)GG_uv'^3\}du. \end{aligned} \tag{8}$$

Therefore, for a family $\mathcal{J}$, we have by (2):

$$v'' = A + Bv' + Cv'^2 + Dv'^3, \tag{9}$$

where

$$\left\{\begin{aligned} A &= (1/2)E_v/G + (E/G)^{1/2}P, \\ B &= (1/2)E_u/E - G_u/G + (E/G)^{1/2}Q, \\ C &= E_v/E - (1/2)G_v/G + (G/E)^{1/2}P, \\ D &= -(1/2)G_u/E + (G/E)^{1/2}Q, \end{aligned}\right. \tag{10}$$

and

$$Q_u - P_v = (k - K)(EG)^{1/2}. \tag{11}$$

That is: *the form* (9) *with additional condition* (11) *is characteristic of curves having the property* $\mathcal{E} = k\mathcal{A}$ *upon the surface S.*

3. **Geodesic representation of the family $\mathcal{J}$.** We now have to impose the further property on the family $\mathcal{J}$.

Since the parametric curves on the surface S_1 form an orthogonal system, the differential equation of geodesics of S_1 is found to be[5]

---

[3] Cf. G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, 1894, p. 47.

[4] Cf. W. Blaschke, *Vorlesungen über Differentialgeometrie*, 3d edition, 1930, p. 175. Write $F=0$, $u'=1$, $u''=0$.

[5] Darboux, loc. cit., p. 49.

$$
(12)\qquad \begin{aligned} v'' = \frac{1}{2}\frac{\partial E_1}{\partial v}\Big/ G_1 + \left(\frac{1}{2}\frac{\partial E_1}{\partial u}\Big/ E_1 - \frac{\partial G_1}{\partial u}\Big/ G_1\right)v' \\ + \left(\frac{\partial E_1}{\partial v}\Big/ E_1 - \frac{1}{2}\frac{\partial G_1}{\partial v}\Big/ G_1\right)v'^2 - \frac{1}{2}\left(\frac{\partial G_1}{\partial u}\Big/ E_1\right)v'^3. \end{aligned}
$$

In order that they should correspond to the curves of the family $\mathcal{J}$ upon S, it is necessary and sufficient that the differential equations (9) and (12) be coincident with each other. This gives

$$
(13)\qquad \left\{ \begin{aligned} &\frac{1}{2}\frac{\partial E}{\partial v}\Big/ G + (E/G)^{1/2}P = \frac{1}{2}\frac{\partial E_1}{\partial v}\Big/ G_1, \\ &\frac{1}{2}\frac{\partial E}{\partial u}\Big/ E - \frac{\partial G}{\partial u}\Big/ G + (E/G)^{1/2}Q = \frac{1}{2}\frac{\partial E_1}{\partial u}\Big/ E_1 - \frac{\partial G_1}{\partial u}\Big/ G_1, \\ &\frac{\partial E}{\partial v}\Big/ E - \frac{1}{2}\frac{\partial G}{\partial v}\Big/ G + (G/E)^{1/2}P = \frac{\partial E_1}{\partial v}\Big/ E_1 - \frac{1}{2}\frac{\partial G_1}{\partial v}\Big/ G_1, \\ &\frac{1}{2}\frac{\partial G}{\partial u}\Big/ E - (G/E)^{1/2}Q = \frac{1}{2}\frac{\partial G_1}{\partial u}\Big/ E_1. \end{aligned} \right.
$$

From the first and the last of these equations we have the expressions for P and Q:

$$
(14)\qquad \left\{ \begin{aligned} P &= \frac{1}{2}(G/E)^{1/2}\left\{\frac{\partial E_1}{\partial v}\Big/ G_1 - \frac{\partial E}{\partial v}\Big/ G\right\}, \\ Q &= \frac{1}{2}(E/G)^{1/2}\left\{\frac{\partial G}{\partial u}\Big/ E - \frac{\partial G_1}{\partial u}\Big/ E_1\right\}. \end{aligned} \right.
$$

Substitution of these expressions in the remaining equations of (13) shows that the fundamental quantities E, G; E_1, G_1 are related by

$$
(15)\qquad \left\{ \begin{aligned} \frac{\partial}{\partial v}\log\frac{EG_1}{GE_1} &= \left(1 - \frac{GE_1}{EG_1}\right)\frac{\partial \log E_1}{\partial v}, \\ \frac{\partial}{\partial u}\log\frac{EG_1}{GE_1} &= \left(\frac{EG_1}{GE_1} - 1\right)\frac{\partial \log G_1}{\partial u}. \end{aligned} \right.
$$

In interpreting these conditions, it is important to distinguish the case $(E/G):(E_1/G_1) = 1$ from $(E/G):(E_1/G_1) \neq 1$. If $(E/G):(E_1/G_1) = 1$, then we can put

$$
(16)\qquad E_1 = \rho E, \qquad G_1 = \rho G,
$$

where $\rho \neq 0$, so that the surfaces S and S_1 are conformal. In this case the equations (15) are satisfied identically, while the equations (14) give

$$\tag{17} \begin{cases} P = \dfrac{1}{2}(E/G)^{1/2}\dfrac{\partial \log \rho}{\partial v}, \\ Q = -\dfrac{1}{2}(G/E)^{1/2}\dfrac{\partial \log \rho}{\partial u}. \end{cases}$$

Substituting these expressions in (11) we have

$$\tag{18} \frac{\partial}{\partial v}\left\{(E/G)^{1/2}\frac{\partial \log \rho}{\partial v}\right\} + \frac{\partial}{\partial u}\left\{(G/E)^{1/2}\frac{\partial \log \rho}{\partial u}\right\} + 2(k - K)(EG)^{1/2} = 0.$$

This condition can be interpreted geometrically as follows: by means of the formula of G. Frobenius for Gaussian curvature[6] we obtain for the surface S with linear element (6)

$$\tag{19} K = -\frac{1}{2W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\},$$

where $W=(EG)^{1/2}$, and similarly, for the surface S_1 with linear element (7)

$$\tag{20} K_1 = -\frac{1}{2W_1}\left\{\frac{\partial}{\partial v}\left(\frac{\partial E_1}{\partial v}\Big/ W_1\right) + \frac{\partial}{\partial u}\left(\frac{\partial G_1}{\partial u}\Big/ W_1\right)\right\},$$

where $W_1=(E_1G_1)^{1/2}$.

If the expressions of E_1 and G_1 given by (16) are substituted in the right-hand member of (20), then

$$K_1 = -\frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\} - \frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}\left((E/G)^{1/2}\frac{\partial \log \rho}{\partial v}\right) + \frac{\partial}{\partial u}\left((G/E)^{1/2}\frac{\partial \log \rho}{\partial u}\right)\right\}.$$

A reference to (18) and (19) gives immediately the relation

$$\tag{21} \rho K_1 = k,$$

which is, of course, equivalent to (18).

If $k \neq 0$, then we have

[6] Blaschke, loc. cit., p. 117.

$$(22) \qquad \begin{cases} ds^2 = (K_1/k)(E_1du^2 + G_1dv^2), \\ ds_1^2 = E_1du^2 + G_1dv^2. \end{cases}$$

Therefore *the surface S_1 may be arbitrarily selected, the only restriction being that it is non-developable, and any surface S, necessarily conformal to S_1, with the linear element*

$$(23) \qquad ds^2 = (K_1/k)ds_1^2$$

possesses the two stated properties, where K_1 is the Gaussian curvature of S_1 and k a constant different from zero. In particular when a surface of constant Gaussian curvature k is taken for S_1, the corresponding surface S is applicable to S_1.

This result not only proves the existence but also furnishes a remarkable class of the surfaces under consideration.

On the contrary, if $k=0$ in the relation (21), then K_1 must necessarily vanish, because ρ is by no means zero, so that *the surface S_1 is developable*. In this case, the surface S is arbitrary and $\mathcal{J}$ must be a conformal image of the ∞^2 straight lines of a plane. That no other family of curves upon a generic surface S can be linear and such that the sum of the angles in every triangle of curves in the family is two right angles has been proved analytically by E. Kasner[7] and synthetically by J. Douglas.[8]

We now consider the case where

$$(24) \qquad (E/G):(E_1/G_1) \neq 1.$$

The partial differential equations (15) are easily integrated, and the result may be written in the form

$$(25) \qquad (EG_1)/(GE_1) = 1 - E_1U^2, \quad (GE_1)/(EG_1) = 1 + G_1V^2,$$

where U denotes any function of u alone, and V any function of v alone. The assumption (24) shows that neither U nor V is zero.

From (25) follows the relation

$$(26) \qquad E_1U^2(G_1V^2 + 2) = G_1V^2(2 - E_1U^2).$$

It may happen that both members of (26) are zero. Since E_1, G_1, U, V are different from zero, we have in this case

---

[7] E. Kasner, *A characteristic property of isothermal systems of curves*, Math. Ann. vol. 59 (1904) pp. 352–354.

[8] J. Douglas, *A criterion for the conformal equivalence of a Riemann space to a Euclidean space*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 299–306.

(27) $$E_1 = 2U^{-2}, \qquad G_1 = -2V^{-2},$$

so that

(28) $$E/G = V^2/U^2.$$

The expressions (14) now become

(29) $$P = -\frac{1}{2}\frac{V}{U}\frac{\partial \log E}{\partial v}, \qquad Q = \frac{1}{2}\frac{U}{V}\frac{\partial \log G}{\partial u}.$$

Substituting them in (18) and remembering the formula (19) in addition to the relation (28), we obtain

(30) $$k = 0.$$

That is, *the sum of the angles in every triangle formed by three curves of the family $\mathcal{J}$ under consideration must be two right angles*, and the linear elements of the surfaces S and S_1 are, after a suitable transformation of the type

(31) $$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

reducible to the form

(32) $$\begin{cases} ds^2 = \lambda(u, v)(du^2 + dv^2), \\ ds_1^2 = du^2 - dv^2. \end{cases}$$

It remains for us to consider the case

(33) $$(2 - E_1U^2)(G_1V^2 + 2) \neq 0.$$

Setting (26) in the form

(34) $$(E_1U^2)/(2 - E_1U^2) = (G_1V^2)(G_1V^2 + 2) = \tau,$$

we obtain

(35) $$E_1 = 2\tau(1 + \tau)^{-1}U^{-2}, \qquad G_1 = 2\tau(1 - \tau)^{-1}V^{-2}$$

and

(36) $$(E/G)^{1/2} = (1 - \tau)(1 + \tau)^{-1}VU^{-1}.$$

The expressions (14) for P and Q may be written in the form

(37) $$\begin{cases} P = \dfrac{1}{2}\left\{\dfrac{V}{U}\dfrac{\partial \log E_1}{\partial v} - \dfrac{1}{W}\dfrac{\partial E}{\partial v}\right\}, \\ Q = -\dfrac{1}{2}\left\{\dfrac{U}{V}\dfrac{\partial \log G_1}{\partial u} - \dfrac{1}{W}\dfrac{\partial G}{\partial u}\right\}. \end{cases}$$

Consequently, we have

$$Q_u - P_v = -\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) + \frac{1}{2}\left\{\frac{\partial}{\partial u}(G_u/W) + \frac{\partial}{\partial v}(E_v/W)\right\},$$

namely,

$$(38)\qquad \begin{aligned} Q_u - P_v = &-\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) \\ &-\frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) - WK, \end{aligned}$$

as may easily be seen on account of (19). By comparison of (38) with (11) it follows that

$$(39)\qquad 2k(EG)^{1/2} = -\frac{1}{V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right).$$

In virtue of (35) we reach the relation:

$$(40)\qquad 2k(EG)^{1/2} = -\frac{1}{UV}\left[\left(U\frac{\partial}{\partial u}\right)^2\log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2\log\frac{\tau}{1+\tau}\right].$$

It is important to distinguish the case $k=0$ from $k\neq 0$. The latter case is of more interest; we find by means of (40) an additional relation between E and G, namely,

$$(41)\qquad (EG)^{1/2} = -(1/2kUV)\left[\left(U\frac{\partial}{\partial u}\right)^2\log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2\log\frac{\tau}{1+\tau}\right].$$

Therefore the linear elements of S and S_1 are, from (35), (36) and (41), given by

$$(42)\qquad \begin{cases} ds^2 = -\dfrac{1}{2k}\left[\left(U\dfrac{\partial}{\partial u}\right)^2\log\dfrac{\tau}{1-\tau} + \left(V\dfrac{\partial}{\partial v}\right)^2\log\dfrac{\tau}{1+\tau}\right] \\ \qquad\cdot\left[\dfrac{1-\tau}{1+\tau}U^{-2}du^2 + \dfrac{1+\tau}{1-\tau}V^{-2}dv^2\right], \\ ds_1^2 = \dfrac{2\tau}{1+\tau}U^{-2}du^2 + \dfrac{2\tau}{1-\tau}V^{-2}dv^2, \end{cases}$$

where τ denotes an arbitrary function of u, v.

If the parameters u, v are subjected to a suitable transformation of type (31), then the pair of surfaces S and S_1 is characterized by the linear elements of a special form:

$$(43)\quad \begin{cases} ds^2 = -\dfrac{1}{2k}\left\{\dfrac{\partial^2}{\partial u^2}\log\dfrac{\tau}{1-\tau}+\dfrac{\partial^2}{\partial v^2}\log\dfrac{\tau}{1+\tau}\right\}\left\{\dfrac{1-\tau}{1+\tau}\,du^2+\dfrac{1+\tau}{1-\tau}\,dv^2\right\}, \\ ds_1^2 = \dfrac{2\tau}{1+\tau}\,du^2+\dfrac{2\tau}{1-\tau}\,dv^2, \end{cases}$$

involving an arbitrary function τ of u and v.

If $k=0$, then (39) becomes, by applying a transformation of type (31),

$$(44)\quad \frac{\partial^2}{\partial v^2}\log E_1 + \frac{\partial^2}{\partial u^2}\log G_1 = 0.$$

Therefore we obtain

$$(45)\quad \begin{cases} ds_1^2 = (E^{-1/2} - G^{-1/2})(E^{1/2}du^2 + G^{1/2}dv^2), \\ ds^2 = Edu^2 + Gdv^2, \end{cases}$$

where the quantities E and G are related by

$$(46)\quad \frac{\partial^2}{\partial v^2}\log\left(1 - (E/G)^{1/2}\right) + \frac{\partial^2}{\partial u^2}\log\left((G/E)^{1/2} - 1\right) = 0.$$

Thus the problem of determining the form of linear elements of a pair of surfaces S and S_1 with the stated properties is completely solved.

It should be observed that the analogous problem for higher dimensional spaces may be of some interest. We hope to consider it in a future paper.

MEITAN, KWEICHOW, CHINA