A NOTE ON DIFFERENTIAL POLYNOMIALS

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The following theorem indicates to what extent the expression of a differential polynomial[1] G as an element of the differential ideal determined by F is unique.

THEOREM I. *Let* $F\neq 0$, $C_0, C_1, \cdots, C_s$ *be differential polynomials in the unknowns* $y_1, \cdots, y_n$ *with coefficients in an abstract differential field* $\mathcal{J}$. *Let* $F^{(i)}$ *be the ith derivative of* F *and let*

$$C_0F + C_1F' + \cdots + C_sF^{(s)} \tag{1}$$

be identically zero. Then each C_i *is in the perfect ideal generated by* F.[2]

We need merely show that any solution $y_j=\bar{y}_j$ $(j=1, \cdots, n)$, in any extension $\mathcal{J}_1$ of $\mathcal{J}$, of the form F is a solution of each C_i.[3] Since this is true if F has no solutions, we may assume that F effectively involves the unknowns. Make the substitution $y_j=z_j+\bar{y}_j$ in (1). Let A consist of the terms of F of lowest degree in the z_j and their derivatives. Collecting terms of the same degree, we see that

$$C_0(\bar{y})A + \cdots + C_s(\bar{y})A^{(s)} = 0, \tag{2}$$

where $C_i(\bar{y})$ is the element of $\mathcal{J}_1$ obtained by substituting $y_j=\bar{y}_j$ $(j=1, \cdots, n)$ in C_i. Let A be of order $p\geqq 0$ in some z_k which it effectively involves, let $z_{k,m}$ be the mth derivative of z_k, and let S be the partial derivative of A with respect to $z_{k,p}$. For $i>0$, $A^{(i)}$ can be written as $Sz_{k,p+i}+B_i$, where B_i is some form of order less than $p+i$ in z_k. Now (2) becomes

$$C_s(\bar{y})Sz_{k,p+s} + D = 0 \tag{3}$$

where D has order less than $p+s$ in z_k. Hence $C_s(\bar{y})=0$. In turn $C_{s-1}, \cdots, C_0$ must vanish for $y_j=\bar{y}_j$ as desired.

Using the ideas of the above proof together with a uniqueness result of J. F. Ritt,[4] one can very easily prove the following generalization.

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[1] For definitions of differential fields, polynomials, and ideals, see H. W. Raudenbush, Ann. of Math. (2) vol. 34 (1933) pp. 509–517.

[2] For a result analogous to Theorem I for ordinary polynomials, see Satz 1 of E. Lasker, *Zur Theorie der Moduln und Ideale*, Math. Ann. vol. 60 (1905) pp. 20–116.

[3] H. W. Raudenbush, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 361–368.

[4] *On singular solutions* $\cdots$, Ann. of Math. vol. 37 (1936) pp. 552–617, §§1–3.

THEOREM II. *Let* $C_1P_1 + \cdots + C_sP_s$ *be identically zero, where the* P_i *are distinct power products each of degree* $d > 0$ *in a nonzero* F *and its derivatives. Then each* C_i *is in the perfect ideal generated by* F.

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ON THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF FORM $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$

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One of the oldest unsolved mathematical problems is the following one: Are there odd perfect numbers?[2] If such a number n exists, it must have the form

$$n = p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}$$

where $p, q_1, q_2, \cdots, q_t$ are primes and $p \equiv \alpha \equiv 1 \pmod 4$. This has been proved by Euler.[3] Sylvester[4] obtained estimates for t, in particular $t \geqq 4$, and $t \geqq 7$ if $n \not\equiv 0 \pmod 3$. Recently, it was shown by R. Steuerwald[5] that the case $\beta_1 = \beta_2 = \cdots = \beta_t = 1$ is impossible, and by H. J. Kanold[6] that the same is true for $\beta_1 = \beta_2 = \cdots = \beta_t = 2$. Moreover Kanold proved that n is not perfect if the greatest common divisor d of $2\beta_1 + 1, 2\beta_2 + 1, \cdots, 2\beta_t + 1$ is divisible by 9, 15, 21, or 33, and some similar results. All these results deal with the case $d > 1$.

In the following, it will be proved that no odd perfect number n of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$ exists. Here we have $d = 1$. For the proof I use

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[1] The relation of the results of this paper to another paper by H. J. Kanold, *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math. vol. 184 (1942) pp. 116–124, will be considered in an addendum to be published in the December Bulletin.

[2] For the history of the problem see Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33.

[3] *Commentationes arithmeticae collectae*, vol. 2, *Tractatus de numerorum doctrina* 1849, p. 514; *Opera postuma*, vol. 1, 1862, pp. 14–15.

[4] *Sur l'impossibilité de l'existence d'un nombre parfait impair qui ne contient pas au moins* 5 *diviseurs premiers distincts*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 522–526; *Collected mathematical papers*, vol. 4, 1912, pp. 611–614. *Sur une classe spéciale des diviseurs de la somme d'une série géométrique*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 446–450; *Collected mathematical papers*, vol. 4, 1912, pp. 607–610.

[5] *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München, 1937, pp. 68–72.

[6] *Untersuchungen über ungerade vollkommene Zahlen*, J. Reine Angew. Math. vol. 183 (1941) pp. 98–109.