

## BOOK REVIEWS

*Analytic topology.* By G. T. Whyburn. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278 pp. \$4.75.

In an effort to place the subject matter of this book within the larger domain covered by the title, we take up two illustrations that are in some sense typical of the contents. The first selection is for those who prefer to think in terms of invariants. One finds many-to-one transformations each of which preserves, a priori, certain topological properties, and a subsequent search for new invariants, rather than a study of isometric or other invariants that might seem to fall within the title. Let  $F$  be a class of transformations and  $S$  be a class of spaces. It is desired to choose  $S$  so that any element of  $S$  maps onto another element of  $S$  under every mapping from  $F$ , and that, furthermore, there be in  $S$  a particular "generator"  $E$ , such that, given any element  $S^*$  of  $S$ , there is an element  $f$  in  $F$  for which  $f(E) = S^*$ . For the case of locally connected continua  $S$ , and the continuous mappings  $F$ , this result is well known; here  $E$  may be taken as an arc. Other examples are: boundary curves  $S$ , simple closed curve  $E$ , and non-alternating transformations  $F$ ; cactoids  $S$ , sphere  $E$ , and monotone mappings  $F$ ; or hemi-cactoids, 2-cell, and monotone mappings. In many instances  $f(S^*) = E$  is also possible for some  $f$  in  $F$ . The composition of two elements of  $F$  is always in  $F$ , at least on spaces from  $S$ . A similar situation in which  $F$  is unknown and  $S$  contains only one element, that is, when the original and image spaces are homeomorphic, is of considerable importance and largely an open question.

The second selection serves to emphasize the highly natural relationship between the older structural ideas and the analytic concepts that form the major part of this work. In order that a continuum  $M$  should be a simple continuous arc it is necessary and sufficient that  $M$  contain at most two points that do not separate  $M$ . This characterization is due to R. L. Moore. By its use of an array of cut points between the two end points, it strongly suggests a powerful technical use of the notion of a cutting for singling out linear arrays in continua. One might say that the theories of dimension and regularity given by Menger and Urysohn are based on a similar, but not the same, approach. When the cuttings are points this technique, roughly speaking, culminates in the cyclic element theory for certain special continua. For non-separated cuttings of more than one point this technique yields other results that are unfortunately not as well

known (Chapter III). It also affords a simple and natural transition from the structural point of view to the analytic in the form of the non-alternating transformation, since this transformation has (in addition to continuity) the property that those counterimages  $f^{-1}(b)$  which are cuttings form a non-separated collection. When considered in conjunction with cyclic elements this mapping acts in a way that suggests a common background with these elements. But the non-alternating transformation also has properties in common with the strongly analytic (Chapter X) interior transformation which preserves open sets. These relations become more evident when non-alternating is specialized to monotone —  $f^{-1}(b)$  connected for all  $b$  in  $B$ . In this connection see quasi-monotone. Finally, as was shown by Moore, the monotone image  $B=f(E)$  of a topological sphere is again a sphere provided no continuum  $f^{-1}(b)$  separates  $E$ . Since the general properties of these mappings are thoroughly treated it is evident that some parts of this book provide a highly appropriate setting for certain results from the work of Moore, Menger and Urysohn. Various other selections would show a strong representation of other mathematicians.

The author, apparently motivated by a desire to select a nucleus of basic material, has left out a considerable portion of his own work, that of his students, and others, particularly work dealing with local separating points and with property  $S$ . There is no extensive use of metric methods, combinatorial topology, nor group theoretic material. After brief study the book permits considerable flexibility, but the various chapters can not be read independently.

Since the contents are self-contained and carefully integrated, this volume will have considerable interest for students and certain possibilities as a text. In this connection the various exercises in the first three chapters deserve mention; but it must also be said that it moves swiftly into the chosen course and does not contain many examples.

Roughly, the purely analytic topology occupies the latter half of the book, but one finds a growing use of analytic methods beginning early (Chapter II). In the following outline by chapters, some notices of errors or omissions are given, together with additional cross references and other comments, but these remarks are neither systematic nor complete.

The first half of Chapter I leads from a set of six axioms to the metrization theorem. It is then assumed that all spaces are separable and metric. Compactness is used in the sense of self-compact (subsets conditionally compact). The student will probably wish to read supplementary material concerning the idea of compactness. A locally

compact and connected set is a generalized continuum. The chapter continues with connectivity for sets and their limits superior, and then takes up continua, the Brouwer reduction theorem, continua of convergence, and local connectivity. The concluding section on property  $S$  and uniformly locally connected sets makes a careful comparison of these important properties. Some of the logical niceties of definition are absent, and on page 4,  $q \subset K$  should be  $q \in K$ .

The second chapter will probably overshadow many other parts of the book as an item of general interest because of its classical results, the interesting nature of the proofs, and the way it gives emphasis to the title of the book. Using the familiar existence-extension technique for uniformly continuous mappings, Whyburn reduces the existence part to a flexible central result which is subsequently used to characterize locally connected continua as continuous images of the unit interval, and compact sets as mappings of the Cantor discontinuum. Furthermore this central result yields an imbedding theorem which, paired with arcwise connectivity for locally connected continua, gives the Moore-Menger arcwise connectivity for complete connected and locally connected sets, and other results. This imbedding theorem furnishes insight into "arcless" continua, but, due to the arrangement of the material, this comment (furnished by the original paper) was omitted here. An altered form of the central result established by Harrold is included, although many of his related results are not. The chapter begins with the necessary preliminaries on transformations, complete sets, and uniform continuity; it concludes with the arcwise connectivity theorem for continua. This final result is established in three ways, including the Kelley proof, so short that it once enjoyed considerable circulation in oral form. The symbols for closure and complete closure (dash and tilde, page 28) could be confused, and the waves over certain letters on page 30 are poorly done. Otherwise the typography is excellent through the whole book.

The fundamental importance of the treatment of non-separated cuttings in Chapter III has been indicated earlier in this review. In addition to the author's order (Menger-Urysohn) theorems for various separating points, the Moore characterizations of the arc and simple closed curve, and the theorem of W. H. Young, there is much basic material on (linear) order, Borel classes, and the nature and existence of cuttings. In particular the set  $E(a, b)$  of all points separating two given points has a natural order, is closed for some continua, and—the author used the Wilder proof—compact in locally connected continua. Various examples might be given; the book has none. In line 5 from bottom of page 41, "points" should be "sets."

That the order is "natural," top of page 43, must be shown.

Chapters IV and V might well have been prefaced by the author's paper, *Concerning maximal sets* (Bull. Amer. Math. Soc. 1934), which does not appear in the bibliography. In every other sense these chapters are a polished presentation of the cyclic element theory, incorporating the recent extension to semi-locally-connected continua, the work of Ayres on cyclic connectedness and the Whyburn contributions to multicoherence (Eilenberg) and the classification of continua among curves. The results in section 7, Chapter IV, are particularly important; see also the theorems at the top of page 107. The lemma page 175, top, should be mentioned on pages 72 and 73. Those readers seeking a casual knowledge of cyclic elements may reach this end by studying dendrites (Chapter V), provided they use the pairings: cyclic element—point; cyclic chain—arc;  $H$ -set—connected subset;  $A$ -set—subcontinuum. On page 73, part (c) of (7.1), omit the vertical bar before  $H$ . The references for Chapter IV are particularly rich in worthwhile material; it would have pleased the reviewer to find the work of Kelley, Harry, and Youngs included in the text. Additional reference: Jones, *Aposyndetic continua and certain boundary problems* (Amer. J. Math. 1941).

The special properties that inhere in continua, their complements, and their boundaries, when the containing space is a plane or a sphere, are developed in Chapter VI. This chapter is fundamental—see, for example, the characterization of the sphere—for readers of the book as a whole, and is second to none in its appeal to the casual reader. It is of interest to note that boundary curves take their name from the property exhibited on p. 107. Separation theorems and accessibility are covered, but all the work on primitive skew curves is omitted.

The theory of semi-continuous decompositions is developed and integrated with that of continuous transformations early in Chapter VII. Certain types of transformations are then defined and characterized. One type,  $f(A) = B$ , with  $f^{-1}(b)$  totally disconnected, called "light" becomes increasingly important in the next few chapters. Semiclosed sets, null collections, and related factorizations constitute the main material. However, the brief treatment of the question of finding an  $f(A) = B$  such that  $A$  and  $B$  are homeomorphic should not be overlooked. In this connection a marginal note on page 135 referring to page 171 will prove valuable. In Chapter VIII retractions, quasimonotone transformations and the relative distance mapping are added to the list. The  $k$  to 1 mappings are noticeably absent; the 0-regular and local homeomorphism are only briefly considered later. This chapter is nevertheless full of material from the heart of the

analytic theory. It is highlighted by the monotone-light factorization theorem, the Wallace quasi-monotone theory (middle ground between interior and monotone), some unpublished work on non-alternating mappings of simple links and  $A$ -sets, and new proofs in the work with the relative distance transformation. See the application page 206. In the references Wallace [4] should be [3].

Chapter IX deals with applications of monotone and non-alternating transformations. It presents a pleasing blend of general results and particular applications which fix and enrich the concepts which are involved. The author adds materially to the interest by pausing to discuss the general characteristics of certain theorems. The last section is entirely new and the work of Morrey and Moore on the various cactoids is treated using new proofs. These invariants together with some striking characterizations of non-alternating transformations on boundary curves form the major part of the chapter. The work of Steenrod and Roberts on the monotone images of two-dimensional manifolds is not included. Correction: last line page 181 Lemma (3.41) should be (3.31).

Chapter X is devoted exclusively to interior transformations: on linear graphs the one-dimensional Betti group maps homomorphically; on simple closed curves the mapping is like  $w = z^k$  on  $|z| = 1$ , or the image is an arc. For the light interior transformation on a compact space, there is an arc in the original set that maps topologically onto a given arc in the image space. This inversion of arcs recalls an open question. However, see the example in the book. The treatment of light interior mappings on a 2-cell virtually establishes the invariance of a 2-cell, but is conceived and used as a central tool in proving the invariance of the 2-manifold property, and the action of these mappings in the small on such manifolds. The manifolds are not required to be compact, and thus this new work further supplants pioneer results of Stoilow. The local analysis— $w = z^n$  on  $|z| \leq 1$  in one instance—is used, together with some new results on local homeomorphisms, to introduce an analysis in the large. This culminates in a simple numerical relationship between the Euler characteristics of the manifolds and leads to a considerable number of examples and applications—closed surfaces retain orientability under the inverse, spheres map onto spheres, projective planes, or 2-cells, and so on. Certain results are then extended to pseudo-manifolds using the relative distance transformation. Some outstanding new items are the proofs by direct analysis of the action of interior mappings. Open questions are: interior (non-light) action on manifolds; the special properties of the monotone factor of an interior transformation.

References to Puckett at end of the previous chapter, and his work on the inversion of local connectivity, are pertinent to this chapter. The latter is not in the bibliography.

The first part of Chapter XI deals with locally connected continua, non-alternating interior transformations and retractions, and the existence of these mappings. The second half of the chapter treats mappings onto the circle using the methods of Eilenberg. A second major type of factorization for transformations arises from the concept "equivalence to one." Each mapping of one simple closed curve into another is homotopic to an interior mapping. Results such as these help to orient the student in the latter part of this chapter. Corrections: "onto" should be "into" page 219, line 6 from the bottom; 0 should be 1 in line 2 of proof (5.1) page 221;  $|t+(1-t)f(0)|$  should be a factor of the right member of the equation in (6.1) page 225, and in the line above,  $f_0(X)$  should be  $f_0(x)$ .

The last chapter opens with theorems concerning the fixed point property and fixed cyclic elements; it continues with characterizations, necessary conditions, and other relations between pointwise almost periodic, regularly almost periodic, pointwise periodic, and periodic properties together with the newly defined Whyburn orbit. The central concepts of this chapter may be traced to many sources, but in general these developments are in the spirit of the work of Ayres. Nearly every page contains unpublished results and evidence of considerable improvement in the methods of proof. The full force of certain combinations of theorems becomes apparent only under systematic study, but the beginner should not find it difficult to progress using the individual theorems and numerous examples. There are numerous corrections. The third line in the proof of (1.11) page 240 should contain  $S$  and not  $X$ ;  $p_{i_n}$  should be  $p_{i_k}$  in line 3, Lemma 2, page 244; and in line 1, §6, page 253,  $\subset$  should be  $=$ . Three others, all on page 225, are: line 1 of (6.2),  $\subset$  should be  $=$ ; lines 8 and 9,  $-n+1$  instead of  $-n$ ; lines 12 and 15,  $n+1$  instead of  $n$ . Finally, page 264, lines 14 and 25,  $z^n$  should be

$$e^{2\pi i/nz} \text{ or } z \exp\left(\frac{2\pi i}{n}\right).$$

This volume is obviously no catalogue of chance items of research. Some of its most striking general features are the new proofs, the integration of the material, and the way it stirs one's interest in innumerable allied topics without violating unity of form and flow of thought. To these virtues may be added a clear pithy style; many theorems are stated in one or two lines. References at the end of each

chapter, a large bibliography, a table of contents and an index, give flexible coverage of all items without being cumbersome. It should enjoy a long life and constant use by all who find interest in this type of work.

G. E. SCHWEIGERT

*Poisson's exponential binomial limit; Table I—Individual Terms; Table II—Cumulated Terms.* By E. C. Molina. New York, Van Nostrand, 1942. viii+46+ii+47 pp. \$2.75.

If  $p$  is the probability of a "success" in a single trial, it is well known that the probability of  $x$  "successes" in  $n$  independent trials is given by

$$(1) \quad C_x^n p^x (1-p)^{n-x}$$

which is the  $(x+1)$ st term in the expansion of the binomial  $[p+(1-p)]^n$ . If the limit of (1) is taken as  $p \rightarrow 0$  and  $n \rightarrow \infty$  in such a way that  $np = a$ , one obtains

$$(2) \quad a^x e^{-a} / x!$$

which is the  $(x+1)$ st term of a distribution originally published by Poisson in 1837. This function not only arises as an approximation to the binomial term (1) for large  $n$  and small  $p$ , but also arises in other problems, as for example in the integration of the chi-square distribution.

Table I of the present book is a tabulation of values of (2) to six places of decimals for  $a = 0.001(0.001)0.01(0.01)0.3(0.1)15(1)100$  and  $x = 0(1)150$ ;  $x$ , of course, being carried far enough for each given value of  $a$  to cover values of (2) to six places of decimals, not all zero. Table II gives the values of  $P(c, a) = \sum_{x=c}^{\infty} a^x e^{-a} / x!$  to six places of decimals for the same range of values of  $a$  and for  $c = 0(1)153$ .

The book has been lithographed by Edwards Brothers and is bound with a flexible paper cover.

Various parts of the tables have appeared in earlier publications. For example, L. v. Bortkiewicz (*Das Gesetz der kleinen Zahlen*, Leipzig, 1898) published tables of (2) to four places of decimals for  $a = 0.1(0.1)10.0$  and  $x = 0(1)24$ . H. E. Soper (*Biometrika*, vol. 10 (1914)) published a table of (2) to six places of decimals for  $a = 0.1(0.1)15.0$  and  $x = 0(1)37$ , which was reprinted in Karl Pearson's *Tables for statisticians and biometricians*, Cambridge, 1914. E. C. Molina (*Amer. Math. Monthly*, 1913) published tables of  $c$  for  $P(c, a) = 0.0001, 0.001$  and  $0.01$ ; for  $a = 0.0001$  to 928, and similar,