A CONJECTURE OF ORE ON CHAINS IN PARTIALLY ORDERED SETS

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In a recent investigation, Ore[1] has given a form of the Jordan-Hölder theorem valid for an arbitrary partially ordered set P. This theorem involves essentially the deformation of one chain into another by successive steps, each step being like that used in the conventional Jordan-Hölder theorem. Ore observes that his first theorem would be slightly easier to apply if it were proved under a weaker hypothesis. The modified theorem runs as follows:[2]

THEOREM. *If P is a partially ordered set in which every chain joining two elements is finite, then any complete chain between two elements $b<a$ can be deformed into any other complete chain between the same two elements.*

The proof rests on this lemma:

LEMMA. *Under the hypothesis of the theorem, if C is a complete chain from b to a which cannot be deformed into the complete chain D from b to a, there exist in P elements $b'<a'$ and complete chains C' and D' from b' to a' such that C' cannot be deformed into D' and such that $b \leqq b'$, $a' \leqq a$ where either $b<b'$ or $a'<a$.*

PROOF. *Case* 1. C and D have in common the element e, $b<e<a$. Then either C_b^e cannot be deformed into D_b^e, or C_e^a cannot be deformed into D_e^a. In these two cases, set $b'=b$, $a'=e$ or $b'=e$, $a'=a$, respectively.

Case 2. C and D have no elements in common. Since C cannot be deformed into D, they cannot together constitute a simple cycle. There will then exist, say, elements c in C and d in D with $b<c<a$, $b<d<a$ and an element m in P with $c \leqq m<a$, $d \leqq m<a$. Because of the hypothesis that every chain in P joining two elements is finite, there will exist in P finite complete chains E_m^a, F_c^m, G_d^m. Then b is joined to a by four complete chains,

$$C_b^c + C_c^a, \qquad C_b^c + F_c^m + E_m^a,$$
$$D_b^d + G_d^m + E_m^a, \qquad D_b^d + D_d^a.$$

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[1] Oystein Ore, *Chains in partially ordered sets*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 558–566.

[2] Terminology and notation follow the paper of Ore.

Since C cannot be deformed into D, one of the following three deformations must be impossible:

$$\overset{a}{C}_c \to \overset{m}{F}_c + \overset{a}{E}_m, \qquad \overset{c}{C}_b + \overset{m}{F}_c \to \overset{d}{D}_b + \overset{m}{G}_d,$$
$$\overset{m}{G}_d + \overset{a}{E}_m \to \overset{a}{D}_d.$$

In the first case we set $a' = a$, $b' = c$; in the second case, $a' = m$, $b' = b$; in the third case $a' = a$, $b' = d$. In each case we have the conclusion of the lemma.

To prove the theorem, suppose that P were to contain two complete chains C and D joining b to a in such wise that C cannot be deformed into D. By induction on n, the lemma gives in P elements $a = a_0 \geqq a_1 \geqq \cdots \geqq a_n$ and $b = b_0 \leqq b_1 \leqq b_n \leqq a_n$ such that for each i either $a_{i-1} > a_i$ or $b_{i-1} < b_i$ $(i = 1, \cdots, n)$, and such that there are complete chains C_n, D_n joining b_n to a_n with C_n not deformable into D_n. This construction can be carried on indefinitely, using the axiom of choice to select at each stage a definite pair a_{n+1}, b_{n+1}. This produces two sequences of elements a_i, b_i with

$$b_0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq \cdots \leqq a_2 \leqq a_1 \leqq a_0.$$

Furthermore, the inequality sign holds an infinite number of times here, so that we obtain an infinite chain joining $b = b_0$ to $a = a_0$, contrary to the hypothesis of the theorem.

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