ON A NEW APPLICATION OF JACOBI POLYNOMIALS IN CONNECTION WITH THE MEAN VALUE THEOREM

E. G. GARNEA

Let us consider the classical theorem of mean value, which states that

$$U(\xi) = \frac{1}{b-a}\int_a^b U(x)dx \tag{1}$$

or

$$F'(\xi) = \frac{1}{b-a}\left[F(b) - F(a)\right] \tag{1'}$$

can be satisfied by at least one value of ξ inside[1] the interval (a, b).

In the general case we can add no further precision concerning the position of the value ξ inside the interval (a, b). But if we consider only functions $U(x)$ belonging to a definite class of functions, we can, sometimes, give a more precise determination for this value ξ. We can, in particular, for some classes of functions, determine intervals (a', b'), concentric to (a, b), with

$$b' - a' = \theta(b - a), \qquad 0 \leqq \theta \leqq 1, \tag{2}$$

and such that (1) holds for at least one value ξ inside (a', b'), for every function $U(x)$ belonging to the class considered and for every interval (a, b) for which the classical mean value theorem holds. The smallest number θ which has the above mentioned property for a given class of functions is called its "contraction factor." It results from this definition that the value of the contraction factor depends only on the class of functions considered and is independent of all other factors, such as the interval (a, b), and so on.

If we replace the equation (1) by (1') and repeat the foregoing literally, we define in exactly the same way the contraction factors for classes of functions $F(x)$.

The existence of a contraction factor for certain classes of functions, particularly for polynomials of a real variable, has been proved by Paul Montel.[2] The value of θ as a function of the degree n of the polynomials considered was found independently and almost at the

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[1] The expression "inside" in this paper means: within or at the ends.

[2] P. Montel, Bull. Soc. Math. France vol. 58 (1930) pp. 105–126. See also D. Pompeiu, Annales Scientifiques de l'Université de Jassy vol. 15 (1929) p. 335.

same time by Tchakaloff[3] and Biernacki.[4] Their results have been generalized since by many authors, among whom we quote Favard,[5] Anghelutza,[6] H. L. Krall[7] and Cioranescu,[8] whose paper is particularly important for us.

Let $U(y_1, y_2, \cdots, y_\nu)$ be an analytical function of ν independent variables, polyharmonical of the order $2m$ and of mean value zero within the hypersphere Σ_ν of its ν-dimensional space. Therefore

$$\Delta^{(2m)} U = 0, \tag{3}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4}$$

where $\Delta^{(2m)}$ is the operator of Laplace applied successively $2m$ times and the integral is extended over the inside of the hypersphere Σ_ν of volume element $d\tau$.

We consider now the system of $2m$ equations with the $2m$ unknowns k_i, x_i:

$$\sum_{i=1}^{m} k_i x_i^{2q} = \frac{\nu}{\nu + 2q}, \qquad q = 0, 1, 2, \cdots, 2m-1, \tag{5}$$

and let x_M be the largest of the solutions x_i of this system. N. Cioranescu proved in his above mentioned paper that x_M is the contraction factor of $U(y_1, y_2, \cdots, y_\nu)$ if:

(a) all x_i are real and inside $(-1, +1)$;

(b) all k_i are positive.

We remember that this statement means: If (3), (4), (a) and (b) are satisfied and R is the radius of Σ_ν, there is at least one point P_0 inside the hypersphere of radius $x_M R$ which makes $U(P_0)=0$.

From Cioranescu's demonstration it follows also that if (a) and (b) are satisfied, but the order of U is odd, so that

$$\Delta^{(2m+1)} U = 0, \tag{3'}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4'}$$

[3] L. Tchakaloff, C. R. Acad. Sci. Paris vol. 192 (1931) p. 32.

[4] M. Biernacki, Bulletin de Mathématiques et de Physique, pures et appliquées de l'École Polytechnique de Bucarest, II Année, no. 3, pp. 164–168.

[5] M Favard, C. R. Acad. Sci. Paris vol. 192 (1931) p. 716.

[6] T. Anghelutza, Mathematica, Cluj vol. 6 (1932) p. 140.

[7] H. L. Krall, *On the mean value theorem*, Amer. Math. Monthly vol. 42 (1935) pp. 604–606.

[8] N. Cioranescu, *Quelques propriétés* · · · , Mathematica, Cluj vol. 9 (1935) pp. 184–193.

the contraction factor is the highest of the values x_i, which are solutions of the system[9] of $2m+2$ equations with $2m+2$ unknowns x_i, k_i:

$$(5') \qquad \sum_{i=1}^{m+1} k_i x_i^{2q} = \frac{\nu}{\nu + 2q}, \qquad x_{m+1} = 0, \qquad q = 0, 1, 2, \cdots, 2m.$$

Here below we shall prove that the conditions (a) and (b) are always satisfied, we shall find the values of the contraction factors connected with the zeros of a sequence of Jacobi (more generally of Tchebycheff) polynomials, and shall see that our results are a generalization of Tchakaloff's and Biernacki's theorems.

Let[10] us put $x_i^2 = u_i$ $(i=1, 2, \cdots, m)$. We observe that the second member of (5) can be written:

$$(6) \qquad \frac{\nu}{\nu + 2q} = \int_0^1 \nu u^{\nu+2q-1} du = \int_0^1 (\nu/2) u^{(\nu/2)-1} u^q du.$$

These are for $q=0, 1, 2, \cdots, 2m-1$ the first m moments of the function

$$\psi(x) = (1/2) \int_0^x \nu u^{(\nu/2)-1} du;$$

therefore, we can write (5) as:

$$(7) \qquad \sum_{i=1}^{m} k_i u_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1.$$

We observe that $\psi(x)$ is monotonically increasing in (0, 1); to such $\psi(x)$ correspond, as is well known,[11] a sequence $\{\phi_m(\nu; x)\}$ of orthonormal[12] polynomials, which, in turn, give rise to a mechanical quadrature formula:[11]

$$(8) \qquad \int_0^1 G_{2m-1}(u) d\psi(u) = \sum_{i=1}^{m} H_i G_{2m-1}(l_i)$$

where $G_{2m-1}(x)$ represents an arbitrary polynomial of degree at most $2m-1$, the abscissas l_i are the zeros of $\phi_m(\nu; x)$, all real, distinct and

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[9] This system is given by Cioranescu in his above mentioned paper for $m \leqq 2$.

[10] I am indebted for this interesting method to Professor J. A. Shohat, whose demonstration I follow very closely.

[11] J. A. Shohat, *Théorie générale des polynômes orthogonaux de Tchebycheff*, Mémorial des Sciences Mathématique, fascicule 66, pp. 8, 15.

[12] In this case they are Jacobi polynomials and coincide, for $\nu=2$, with Legendre's polynomials.

inside (0, 1), and the coefficients H_i are all positive. It follows, for $G_{2m-1}(u)=u^q$,

$$(9) \qquad \sum_{i=1}^{m} H_i l_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1.$$

Comparing (7) with (9) we see that a solution of (7) is given by

$$(10) \qquad u_i = l_i, \qquad k_i = H_i,$$

with $0<l_i<1$, $H_i>0$, and, returning to (5):

$$(11) \qquad x_i = \pm (l_i)^{1/2}, \qquad k_i = H_i, \qquad i = 1, 2, 3, \cdots, m.$$

Similar considerations on the m equations 2 to $m+1$ of (5′) (putting for instance $k_i x_i^2 = k_i'$) led also to m values $x_i^2 = l_i'$ $(i=1, 2, \cdots, m)$ to which we have only to add $x_{m+1}=0$, in order to have the complete set.

We may remark that if in $\Delta^{(n)}U=0$, $n=2m$, the moments of $\psi(x)$ are $\alpha_p=\nu/(\nu+2p)$ and the corresponding Jacobi polynomials form a complete sequence; if $n=2m-1$, the moments are $\alpha_p=\nu/(\nu+2(p+1))$ and the corresponding Jacobi polynomials form another complete sequence.

It follows that the systems (5) and (5′) have solutions, where all x_i are real and in absolute value less than one, and all k_i are positive; therefore, the conditions of Cioranescu are always satisfied.[13]

We could find the x_i also solving (5) by the method of Sylvester,[14] which, however, requires lengthy considerations of determinants. Its result is that the solutions x_i of (5), are the zeros of the polynomial:

$$(12) \qquad E_{2m}(\nu; x) = \begin{vmatrix} \frac{x^2}{\nu}-\frac{1}{\nu+2} & \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \cdots & \frac{x^2}{\nu+2(m-1)}-\frac{1}{\nu+2m} \\ \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \cdots & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{x^2}{\nu+2(m-1)}-\frac{1}{\nu+2m} & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} & \cdots & \frac{x^2}{\nu+2(2m-2)}-\frac{1}{\nu+2(2m-1)} \end{vmatrix}.$$

[13] The present method has many points in common with that of J. A. Shohat, *On a certain formula of mechanical quadratures with non-equidistant ordinates*, Trans Amer. Math. Soc. vol. 31 (1929) pp. 449–450.

[14] For Sylvester's method see for instance T. Muir, *Theory of determinants*, vol. II, pp. 332–335. It may be noted that the formula (in Sylvester's notation): $a_{n+1}-a_n\sum\lambda_1+a_{n-1}\sum\lambda_1\lambda_2-\cdots=0$, which we meet there, generalizes an analogous equation indicated by P. Montel in his quoted paper.

Similarly, the solutions of (5′) are the zeros of

$$
(12') \quad E_{2m+1}(\nu; x) = x \begin{vmatrix} \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \cdots & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} \\ \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \frac{x^2}{\nu+6}-\frac{1}{\nu+8} & \cdots & \frac{x^2}{\nu+2(m+1)}-\frac{1}{\nu+2(m+2)} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} & \frac{x^2}{\nu+2(m+1)}-\frac{1}{\nu+2(m+2)} & \cdots & \frac{x^2}{\nu+2(2m-1)}-\frac{1}{\nu+4m} \end{vmatrix}.
$$

These formulae have already been found for $\nu=1$ by Biernacki.[15]

We can thus make the general statement:

If the analytical function $U(P)$ of ν independent variables $y_1, y_2, \cdots, y_\nu$ satisfies the conditions $\Delta^{(n)}U=0$ and $\int_{\Sigma_\nu} U d\tau=0$, there is inside the hypersphere of radius $x_M R$ at least one point P_0 which makes $U(P_0)=0$, R being the radius of Σ_ν and $x_M<1$ the square root of the highest zero of the Jacobi polynomial[16] $\phi_m(\nu; x)$, where $m=n/2$ or $m=(n-1)/2$, according to the parity of n; the same value x_M is also the highest zero of the polynomial[17] $E_n(\nu; x)$.

Taking into account the formulae of Tchebycheff's polynomials:[18]

$$
(13) \qquad \Phi_n(x) = \frac{1}{\Delta_n(\psi)} \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_n \\ \alpha_1 & \alpha_2 \cdots \alpha_{n+1} \\ \cdot & \cdot \;\cdots\; \cdot \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-1} \\ 1 & x \cdots x^n \end{vmatrix}
$$

$$
= x^n - \delta_{n,n-1}x^{n-1} + \delta_{n,n-2}x^{n-2} - \cdots
$$

with

$$
(14) \qquad \Delta_n(\psi) = \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_{n-1} \\ \alpha_1 & \alpha_2 \cdots \alpha_n \\ \cdot & \cdot \;\cdots\; \cdot \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-2} \end{vmatrix}
$$

and the formulae of Szegö:[19]

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[15] M. Biernacki, loc. cit. p. 166.

[16] We can write immediately $\phi_m(\nu; x)$, knowing the moments of $\psi(x)$.

[17] As we shall see, the polynomials $E_n(\nu; x)$ form a *single* sequence of Tchebycheff polynomials, without distinction as to whether n is even or odd.

[18] J. A. Shohat, loc. cit. pp. 3–5.

[19] G. Szegö, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören,* Math. Zeit. vol. 9 (1921) p. 218.

$$(15)\qquad \Delta_{2m}(\psi)\,\Phi_{2m}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & \alpha_{2m} \\ 0 & \alpha_2 & \cdots & 0 \\ \alpha_2 & 0 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \alpha_{2m} & \cdots & 0 \\ 1 & x & \cdots & x^{2m} \end{vmatrix} = \begin{vmatrix} +\alpha_0 x & -\alpha_2 & \cdots & -\alpha_{2m} \\ -\alpha_2 & +\alpha_2 x & \cdots & +\alpha_{2m} x \\ -\alpha_2 x & +\alpha_4 & \cdots & +\alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ +\alpha_{2m-2} x & -\alpha_{2m} & \cdots & -\alpha_{4m-2} \\ -\alpha_{2m} & +\alpha_{2m} x & \cdots & +\alpha_{4m-2} x \end{vmatrix},$$

$$(15')\qquad \Delta_{2m+1}(\psi)\,\Phi_{2m+1}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & \alpha_{2m+2} \\ \alpha_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & 0 & \cdots & 0 \\ 1 & x & \cdots & x^{2m+1} \end{vmatrix} = \begin{vmatrix} +\alpha_0 x & -\alpha_2 & \cdots & +\alpha_{2m} x \\ -\alpha_2 & +\alpha_2 x & \cdots & -\alpha_{2m+2} \\ +\alpha_2 x & -\alpha_4 & \cdots & +\alpha_{2m+2} x \\ \cdot & \cdot & \cdots & \cdot \\ -\alpha_{2m} & +\alpha_{2m} x & \cdots & -\alpha_{4m} \\ +\alpha_{2m} x & -\alpha_{2m+2} & \cdots & +\alpha_{4m} x \end{vmatrix}$$

we find easily:

$$(16)\qquad \begin{aligned} \Delta_{2m}(\psi)\,\Phi_{2m}(x) &= \begin{vmatrix} \alpha_0 & 0 & \cdots & \alpha_{2m} \\ 0 & \alpha_2 & \cdots & 0 \\ \alpha_2 & 0 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \alpha_{2m} & \cdots & 0 \\ 1 & x & \cdots & x^{2m} \end{vmatrix} \\ &= C_{2m+1} \begin{vmatrix} \alpha_0 x^2 - \alpha_2 & \alpha_2 x^2 - \alpha_4 & \cdots & \alpha_{2m-2} x^2 - \alpha_{2m} \\ \alpha_2 x^2 - \alpha_4 & \alpha_4 x^2 - \alpha_6 & \cdots & \alpha_{2m} x^2 - \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m-2} x^2 - \alpha_{2m} & \alpha_{2m} x^2 - \alpha_{2m+2} & \cdots & \alpha_{4m-4} x^2 - \alpha_{4m-2} \end{vmatrix}, \end{aligned}$$

$$(16')\qquad \begin{aligned} \Delta_{2m+1}(\psi)\,\Phi_{2m+1}(x) &= \begin{vmatrix} \alpha_0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & \alpha_{2m+2} \\ \alpha_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & 0 & \cdots & 0 \\ 1 & x & \cdots & x^{2m+1} \end{vmatrix} \\ &= C_{2m+2}\, x \begin{vmatrix} \alpha_2 x^2 - \alpha_4 & \alpha_4 x^2 - \alpha_6 & \cdots & \alpha_{2m} x^2 - \alpha_{2m+2} \\ \alpha_4 x^2 - \alpha_6 & \alpha_6 x^2 - \alpha_8 & \cdots & \alpha_{2m+2} x^2 - \alpha_{2m+4} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} x^2 - \alpha_{2m+2} & \alpha_{2m+2} x^2 - \alpha_{2m+4} & \cdots & \alpha_{4m-2} x^2 - \alpha_{4m} \end{vmatrix} \end{aligned}$$

where

$$(17)\qquad C_{2m} = \begin{vmatrix} \alpha_0 & \alpha_2 & \cdots & \alpha_{2m-2} \\ \alpha_2 & \alpha_4 & \cdots & \alpha_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m-2} & \alpha_{2m} & \cdots & \alpha_{4m-4} \end{vmatrix},$$

$$(17)\qquad C_{2m+1} = \begin{vmatrix} \alpha_2 & \alpha_4 & \cdots & \alpha_{2m} \\ \alpha_4 & \alpha_6 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & \alpha_{2m+2} & \cdots & \alpha_{4m-2} \end{vmatrix}.$$

Equating in (16) the coefficients of x^{2m}, and in (16′) the coefficients of x^{2m+1}, we obtain,[20] taking into account (13):

$$\Delta_{2m}(\psi) = C_{2m}C_{2m+1}$$

and

$$\Delta_{2m+1}(\psi) = C_{2m+1}C_{2m+2}$$

or generally

$$(18)\qquad \Delta_n(\psi) = C_nC_{n+1}.$$

Let us put now:

$$(19)\qquad \alpha_{2m} = \frac{1}{\nu + 2m}, \qquad \alpha_{2m+1} = 0.$$

Then the C_n, easily calculable, being all positive, (18) shows that $\Delta_n(\psi) > 0$. Therefore, the conditions of Hamburger[21] being satisfied, there is a monotonically increasing function $\Psi(x)$[22], whose moments are (19). Comparing (12), (12′) with (16), (16′), we see that the polynomials $E_n(\nu; x)$ are just the sequence of polynomials of Tchebycheff corresponding to $\Psi(x)$. As they form a sequence of Sturm[23] and as $E_n(\nu; 1) > 0$, $E_{2m}(\nu; 0) = (-1)^m C_{2m+1}$, $\lim_{x=0} (1/x)E_{2m+1}(\nu; x) = (-1)^m D_{2m}$, with D_{2m} an easily calculable positive determinant, we find also in this way that all the x_i, zeros of $E_n(\nu; x)$ are real, distinct and inside $(-1, +1)$.[24]

The function U, polyharmonical of order n, can be in particular a polynomial of degree $2n-1$, of ν independent variables. If a polynomial is of an even degree, it may be considered for our purpose as

[20] I have learned that the same formulae have been derived by C. Rees in his thesis on *Elliptic orthogonal polynomials.*

[21] H. Hamburger, *Über eine Erweiterung des Stieltjeschen Momentenproblems,* Math. Ann. vol. 81 (1920) pp. 235–319; vol. 82 (1921) pp. 120–164 and pp. 168–187.

[22] Taking into account the further conditions of the problem, we easily find this function, which is: $d\Psi(x) = (1/2)|x|^{\nu-1}dx$ for $|x| \leqq 1$, $d\Psi(x) = 0$ for $|x| > 1$.

[23] See also Brioschi, *Théorie des déterminants,* pp. 85–86.

[24] By solving completely the system (5) it can be proved also that $k_1 > 0$, but this requires a very long calculation, whereas the ingenious method of J. A. Shohat yields immediately the result. Some rather complicated expressions of the k_i in function of ν are given, for small values of n, in the quoted paper of N. Cioranescu in Mathematica, Cluj vol. 9 (1935) pp. 191–192.

of the next higher odd degree, with its first coefficient equal to zero.[25]

For $\nu=1$ we fall back on the known theorems of P. Montel, Tchakaloff[26] and Biernacki.[27] In fact, the formulae (12), (12′), with (16), (16′) and (19) show us [28] that for $\nu=1$, $E_n(1;\, x)$ are but the polynomials $P_n(x)$ of Legendre; moreover:

$$\Delta^{(n)} U(y) = \frac{d^{2n} U(y)}{dy^{2n}} = 0 \tag{3''}$$

if the degree of U is at most $2n-1$. Applying our results from above to this case we find that if x_M is the highest zero of Legendre's polynomial $P_n(x)$ and

$$\int_a^b U(y)dy = 0 \tag{4''}$$

where $U(y)$ is a polynomial of degree at most $2n-1$, then $U(y)$ has at least one zero inside the interval $(1/2)(b+a) \pm x_M(b-a)/2$. These are precisely the results of Tchakaloff and Biernacki and our main statement from p. 545 can be considered as a generalization of the precisions which the above mentioned authors have brought to the mean value theorem.

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[25] This has been shown for $\nu=1$ by P. Montel in his quoted paper. A slight difference can be noted between his statement and ours, because P. Montel's refers not to the function $U(y)$, but to $F(y)=\int_{y_0}^{y} U(y)dy$.

[26] Tchakaloff, loc. cit. p. 34.

[27] Biernacki, loc. cit. p. 167–168.

[28] We may remark that the formulae (13), (13′), (16) and (16′) attest the equivalence of the solutions found by the two authors in the problem of the contraction factor, in spite of their different form.