THE NUMBER OF INDEPENDENT COMPONENTS OF THE TENSORS OF GIVEN SYMMETRY TYPE

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Let $T_{i_1 \cdots i_p}$ be an arbitrary covariant tensor with respect to an n-dimensional coordinate system, and let

$$(1) \qquad T_{i_1 \cdots i_p} = {}_{[p]}T_{i_1 \cdots i_p} + \cdots + {}_{[\alpha]}T_{i_1 \cdots i_p} + \cdots + {}_{[1^p]}T_{i_1 \cdots i_p}$$

represent the decomposition[1,2] of $T_{i_1} \cdots {}_{i_p}$ into tensors of various symmetry types, the tensor ${}_{[\alpha]}T_{i_1} \cdots {}_{i_p}$ corresponding to the partition $[\alpha]$ of the indices $i_1 \cdots i_p$. The number of independent (scalar) components of $T_{i_1} \cdots {}_{i_p}$ is n^p; and if c_α denotes the number of components of ${}_{[\alpha]}T_{i_1} \cdots {}_{i_p}$, then

$$(2) \qquad n^p = c_{[p]} + \cdots + c_{[\alpha]} + \cdots + c_{[1^p]} = \sum c_\alpha.$$

For $p=2, 3, 4$, J. A. Schouten[3] has obtained expressions for the c_α's in terms of n; but the difficulties of his method become great for larger values of p. The purpose of this paper is to present a method of obtaining c_α in terms of n from the character table for the symmetric group on p letters.

Associated with the immanant tensor[2] $I_{(j)}^{(i)} \equiv {}_{[\alpha]}I_{j_1 \cdots j_p}^{i_1 \cdots i_p}$ we have defined the numerical invariant $r = r_\alpha$, the rank[4] of $I_{(j)}^{(i)}$, which is the greatest integer r for which the tensor

$$(3) \qquad I_{(j_1) \cdots (j_r)}^{(i_1) \cdots (i_r)} = \begin{vmatrix} I_{(j_1)}^{(i_1)} & \cdots & I_{(j_r)}^{(i_1)} \\ \cdot & \cdots & \cdot \\ I_{(j_1)}^{(i_r)} & \cdots & I_{(j_r)}^{(i_r)} \end{vmatrix}$$

does not vanish; here $(i_\lambda) = i_{\lambda 1} \cdots i_{\lambda p}$. For convenience, let us regard $I_{(j)}^{(i)}$, for each (i), as a vector $V_{(j)}$ in $N = n^r$ dimensions. Then from the above definition, it is clear that exactly r_α of the N vectors $V_{(j)}$ are linearly independent. Since ${}_{[\alpha]}T_{(j)} \equiv {}_{[\alpha]}T_{j_1} \cdots {}_{j_p}$ may be defined by

$$(4) \qquad {}_{[\alpha]}T_{(j)} = {}_{[\alpha]}I_{(j)}^{(l)} T_{(l)};$$

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[1] H. Weyl, *The classical groups*, Princeton, 1939, chap. IV.

[2] T. L. Wade, *Tensor algebra and Young's symmetry operators*, Amer. J. Math. vol. 63 (1941) pp. 645–657.

[3] J. A. Schouten, *Der Ricci-Kalkul*, Berlin, 1924, chap. VII.

[4] Richard H. Bruck and T. L. Wade, *Bisymmetric tensor algebra*, II, Amer. J. Math. vol. 64 (1942) pp. 734–753. We shall refer to this paper as B.T.A.II.

exactly r_α of the components of $T_{(j)}$ are linearly independent; thus $c_\alpha \leqq r_\alpha$. But as an alternative way of writing equation (32), B.T.A. II,

$$n^p = r_{[p]} + \cdots + r_{[\alpha]} + \cdots + r_{[1^p]} = \sum r_\alpha. \tag{5}$$

Hence, using (2),

$$n^p = \sum c_\alpha \leqq \sum r_\alpha = n^p, \tag{6}$$

and since the numbers c_α, r_α are non-negative we conclude this fact.

THEOREM I. $c_\alpha = r_\alpha$.

Combining Theorem I with Theorem V, B.T.A. II, we obtain this theorem.

THEOREM II.

$$c_\alpha = \frac{f_\alpha}{p!} \sum_{(\rho)} \chi_\alpha^{(\rho)} \cdot \nu_\rho n^{k_\rho},$$

where $\chi_\alpha^{(\rho)}$ *is the characteristic for class* (ρ) *corresponding to the irreducible representation* $[\alpha]$ *of the symmetric group on* p *letters,*

f_α *is the characteristic corresponding to the class* (1^p),

ν_ρ *is the order of class* (ρ),

$k_\rho = \rho_1 + \rho_2 + \cdots + \rho_p$, *where* $\rho = (1^{\rho_1}, 2^{\rho_2}, \cdots, p^{\rho_p})$.

Another method of finding c_α is given by G. de B. Robinson[5] in relating r_α to A. Young's substitutional analysis.

For $p = 4$ the character table is,[6] with the additional row of values of k_ρ inserted:

Class:	(ρ)	(1^4)	$(1^2, 2)$	$(1, 3)$	(4)	(2^2)
Order:	ν_ρ	1	6	8	6	3
	k_ρ	4	3	2	1	2
	$[4]$	1	1	1	1	1
	$[3, 1]$	3	1	0	-1	-1
	$[2^2]$	2	0	-1	0	2
	$[2, 1^2]$	3	-1	0	1	-1
	$[1^4]$	1	-1	1	-1	1

From this, using Theorem II, we have, for example

[5] G. de B. Robinson, *Note on a paper by R. H. Bruck and T. L. Wade*, Amer. J. Math. vol. 64 (1942) p. 753.

[6] D. E. Littlewood, *Theory of group characters and matrix representations of groups*, Oxford, 1940, p. 265.

$$c_{[4]} = (1/4!)\{1\cdot 1\cdot n^4 + 1\cdot 6\cdot n^3 + 1\cdot 8\cdot n^2 + 1\cdot 6\cdot n + 1\cdot 3\cdot n^2\}$$
$$= C_{n+3,4},$$

and

$$c_{[3,1]} = (3/4!)\{3\cdot 1\cdot n^4 + 1\cdot 6\cdot n^3 + 0\cdot 8\cdot n^2 - 1\cdot 6\cdot n - 1\cdot 3\cdot n^2\}$$
$$= 9C_{n+2,4}.$$

In this manner we obtain the following tables of c_α:

Three-indexed tensors

$[\alpha]$	$[3]$	$[2, 1]$	$[1^3]$
c_α	$C_{n+2,3}$	$4C_{n+1,3}$	$C_{n,3}$

Four-indexed tensors

$[\alpha]$	$[4]$	$[3, 1]$	$[2^2]$	$[2, 1^2]$	$[1^4]$
c_α	$C_{n+3,4}$	$9C_{n+2,4}$	$nC_{n+1,3}$	$9C_{n+1,4}$	$C_{n,4}$

Five-indexed tensors

$[\alpha]$	$[5]$	$[4, 1]$	$[3, 2]$	$[3, 1^2]$	$[2^2, 1]$	$[2, 1^3]$	$[1^5]$
c_α	$C_{n+4,5}$	$16C_{n+3,5}$	$5nC_{n+2,4}$	$36C_{n+2,5}$	$5nC_{n+1,4}$	$16C_{n+1,5}$	$C_{n,5}$

Six-indexed tensors

$[\alpha]$	$[6]$	$[5, 1]$	$[4, 2]$	$[4, 1^2]$	$[3^2]$	$[3, 2, 1]$
c_α	$C_{n+5,6}$	$25C_{n+4,6}$	$(27n/2)C_{n+3,5}$	$100C_{n+3,6}$	$(5/3)C_{n+2,4}C_{n+1,2}$	$(128n/3)C_{n+2,5}$

$[\alpha]$	$[3, 1^3]$	$[2^3]$	$[2^2, 1^2]$	$[2, 1^4]$	$[1^6]$
c_α	$100C_{n+2,6}$	$(5/3)C_{n+1,4}C_{n,2}$	$(27n/2)C_{n+1,5}$	$25C_{n+1,6}$	$C_{n,6}$

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