ON LINEAR COMBINATIONS OF QUADRATIC FORMS

LLOYD L. DINES

The characteristics of linear combinations $\sum \lambda_i Q_i(x)$ of a given set of real quadratic forms

$$(1) \qquad Q_i(x) \equiv \sum_{k,l=1}^{n} \overset{i}{a}_{kl} x_k x_l, \qquad i = 1, 2, \cdots, m,$$

have been considered in several recent papers.[1]

One of the theorems in my earlier paper may be stated as follows:

A necessary and sufficient condition that there exist a linear combination $\sum \lambda_i Q_i(x)$ which is positive definite is that there exist no set of points $x^j = (x_1^j, x_2^j, \cdots, x_n^j) \neq (0, \cdots, 0)$ $(j = 1, 2, \cdots, r)$ such that

$$(2) \qquad \sum_{j=1}^{r} \mu_j Q_i(x^j) = 0, \qquad i = 1, 2, \cdots, m,$$

the coefficients μ_j being positive.

Shortly after the publication of this paper, Fritz John kindly called my attention to the fact that a closely related result is contained in an earlier paper of his.[2]

Certainly John's paper contains essentially the "sufficiency" half of the theorem quoted above. Furthermore it introduces a very interesting suggestion in noting that the validity of relations (2) implies the existence of a quadratic form

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l$$

which is definite or semi-definite, and such that

$$\sum_{k,l=1}^{n} \overset{i}{a}_{kl} \cdot b_{kl} = 0, \qquad i = 1, 2, \cdots, m.$$

[1] Finsler, *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Comment. Math. Helv. vol. 9 (1937) pp. 188–192. Hestenes and McShane, *A theorem on quadratic forms and its application in the calculus of variations*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 501–512. Dines, *On the mapping of n quadratic forms*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 467–471.

[2] *A note on the maximum principle for elliptic differential equations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 268–271.

This condition, unlike those occurring in other treatments of the problem, focuses attention on the coefficients of the given forms. Because of its purely algebraic character and its possible usefulness in applications it seems worthy of further study, and that is the purpose of the present note. In Theorem I there is an analysis of possibilities in terms of this relationship, and in Theorem II there is an equivalent statement from a different and possibly interesting point of view.

1. **Orthogonality of quadratic forms as to their coefficients.** The bilinear form consisting of the sum of products of corresponding coefficients of two quadratic forms

$$Q(x) \equiv \sum_{k,l=1}^{n} a_{kl} x_k x_l, \quad B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l, \quad a_{kl} = a_{lk}, \ b_{kl} = b_{lk},$$

will be denoted by $(Q \cdot B)$. That is[3]

$$(Q \cdot B) = \sum_{k,l=1}^{n} a_{kl} \cdot b_{kl}.$$

Obviously $(Q \cdot B) = (B \cdot Q)$; and for a set of forms such as those in (1), $(\sum \lambda_i Q_i \cdot B) = \sum \lambda_i (Q_i \cdot B)$. Also it may be easily verified that $(Q \cdot B)$ is invariant under the group of orthogonal transformations on $x_1, x_2, \cdots, x_n$.

If the two quadratic forms $Q(x)$ and $B(x)$ satisfy the relation $(Q \cdot B) = 0$ they will be said to be *orthogonal as to coefficients* (or *c-orthogonal*).[4]

Relative to a given system of real quadratic forms (1), we now state

THEOREM I. (a) *A necessary and sufficient condition that there exist a definite linear combination $\sum \lambda_i Q_i$ is that every quadratic form c-orthogonal to all the Q_i be indefinite.*

(b) *A necessary and sufficient condition that every linear combination $\sum \lambda_i Q_i$ be indefinite is that there exist a definite quadratic form c-orthogonal to all the Q_i.*

(c) *A necessary and sufficient condition that there exist a semi-definite*

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[3] M. R. Hestenes has kindly called attention to the fact that this function of the coefficients of two forms was used by L. Fejér in a paper entitled *Über die Eindeutigkeit der Lösung der linearen partiellen Differentialgleichung zweiter Ordnung*, Math. Zeit. vol. 1 (1918) pp. 70–73. Fejér obtained a theorem which may be stated in our notation as follows: *If $Q(x)$ and $B(x)$ are both non-negative, then $(Q \cdot B) \geq 0$.*

[4] However, to avoid bothersome trivialities, we shall assume that an assertion $(Q \cdot B) = 0$ implies *per se* that each of the forms $Q \cdot B$ has at least one nonzero coefficient. In the same spirit, the notation $\sum \lambda_i Q_i$ implies *per se* that at least one λ_i is different from zero.

(but no definite) linear combination $\sum \lambda_i Q_i$ *is that there exist a semi-definite (but no definite) quadratic form c-orthogonal to all the* Q_i.

2. **Proof of Theorem I(a).** In view of the theorem quoted in our introduction, the validity of I(a) is an immediate consequence of the following lemma.

LEMMA. *A necessary and sufficient condition that the quadratic forms* (1) *admit relations* (2) *is that there exist a definite (or semi-definite) form* $B(x)$ *such that*

$$(3) \qquad\qquad (Q_i \cdot B) = 0, \qquad\qquad i = 1, 2, \cdots, m.$$

The first half of this lemma is due to John, as is the following neat proof of it.

Suppose there exist relations (2). These may be written

$$\sum_{j=1}^{r} \mu_j \sum_{k,l=1}^{n} a_{kl}^{i} x_k^{j} x_l^{j} = 0, \qquad i = 1, 2, \cdots, m,$$

or

$$\sum_{k,l=1}^{n} a_{kl}^{i} \cdot b_{kl} = 0 \quad where \quad b_{kl} = \sum_{j=1}^{r} \mu_j x_k^{j} x_l^{j}.$$

Hence the quadratic form $B(x)$ with coefficients b_{kl} is c-orthogonal to every $Q_i(x)$.

Furthermore the form $B(x)$ is definite or semi-definite, since

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l = \sum_{k,l=1}^{n} \left(\sum_{j=1}^{r} \mu_j x_k^{j} x_l^{j} \right) x_k x_l = \sum_{j=1}^{r} \mu_j \left(\sum_{k=1}^{n} x_k^{j} x_k \right)^2.$$

Conversely, suppose there is a definite (or semi-definite) form B satisfying (3). From the assumption of definiteness, it can be expressed in the form

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l = \sum_{j=1}^{r} \mu_j \left(\sum_{k=1}^{n} \alpha_k^{j} x_k \right)^2$$

where the coefficients α_k^{j} are suitably chosen constants and the μ_j are positive constants. Hence

$$b_{kl} = \sum_{j=1}^{r} \mu_j \alpha_k^{j} \alpha_l^{j}.$$

And in view of (3)

$$\sum_{k,l=1}^{n} a_{kl}^{i} \sum_{j=1}^{r} \mu_j \alpha_k^{j} \alpha_l^{j} = 0, \qquad i = 1, 2, \cdots, m,$$

and equivalently

$$\sum_{j=1}^{r} \mu_j \sum_{k,l=1}^{n} a_{kl}^{i} \alpha_k^{j} \alpha_l^{j} = 0, \qquad i = 1, 2, \cdots, m,$$

whence follows (2) with $x_k^{j} = \alpha_k^{j}$.

3. Proof of Theorem I(b). This proof is facilitated by the following corollary.

COROLLARY OF I(a). *If two quadratic forms are c-orthogonal and one is definite, then the other is indefinite.*

The *sufficiency* part of I(b) follows immediately. For if a definite form B is c-orthogonal to every Q_i it is c-orthogonal to every linear combination $\sum \lambda_i Q_i$, and hence by the corollary every such linear combination is indefinite.

To prove the *necessity* part of I(b), we assume that every linear combination $\sum \lambda_i Q_i(x)$ changes sign. On account of the homogeneity property $Q(tx) = t^2 Q(x)$, we are indeed justified in assuming that every such linear combination changes sign as x varies on the unit hypersphere $\|x\| = 1$ in the n-dimensional x-space.

The set of points $\mathfrak{M}_1$ with coordinates represented by

$$\mathfrak{M}_1: \qquad (Q_1(x), Q_2(x), \cdots, Q_m(x)), \qquad \|x\| = 1,$$

in m-dimensional space is closed and bounded. Hence, in view of our assumption, its convex extension $C(\mathfrak{M}_1)$ contains the origin of that m-dimensional space as an *inner* point.[5] This origin can therefore be the centroid of positive masses at a suitably chosen finite and *truly* m-dimensional subset of points[6]

$$\mathfrak{M}_2: \qquad (Q_1(x^p), Q_2(x^p), \cdots, Q_m(x^p)), \qquad p = 1, 2, \cdots, r.$$

To the finite set $\mathfrak{M}_2$ we arbitrarily adjoin the set of n points

$$\mathfrak{M}_3: \qquad (Q_1(x^j), Q_2(x^j), \cdots, Q_m(x^j)), \qquad j = 1, 2, \cdots, n,$$

where each n-partite number $x^j = (x_1^j, x_2^j, \cdots, x_n^j)$ has all zero components with the exception that $x_j^j = 1$.

The set $\mathfrak{N} \equiv \mathfrak{M}_2 + \mathfrak{M}_3$ is a truly m-dimensional finite set, and its convex extension contains the origin as an inner point. Hence this origin can be the centroid of positive masses at *all*[7] points of $\mathfrak{N}$; and the analytic expression of this fact leads to the system of relations

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[5] Cf. Dines, *Convex extension and linear equalities*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 357 Theorem 3.

[6] Ibid. p. 358.

[7] Ibid. p. 358 Theorem 4.

$$(4) \qquad \sum_{j=1}^{n} \mu_j Q_i(x^j) + \sum_{p=1}^{r} \nu_p Q_i(x^p) = 0, \qquad i = 1, 2, \cdots, m,$$

where each μ_j and ν_p is positive.

Applying to the relations (4) precisely the same type of argument as was applied to relations (2) in proving the lemma of §2, we arrive at the existence of a quadratic form $B(x)$ which is c-orthogonal to all the Q_i, which has the formal expression

$$B(x) = \sum_{j=1}^{n} \mu_j x_j^2 + \sum_{p=1}^{r} \nu_p \left(\sum_{k=1}^{n} x_k^p x_k \right)^2$$

and is therefore certainly definite. This completes the proof of I(b).

4. Proof of Theorem I(c). This follows almost immediately from I(a) and I(b). First suppose there is a $\sum \lambda_i Q_i$ which is semi-definite, but none which is definite. Then by I(b) there is no definite form c-orthogonal to all Q_i, and so by I(a) there must be one which is semi-definite.

Conversely, suppose there exists a semi-definite (but no definite) form c-orthogonal to all the Q_i. Then by I(a) there exists no definite linear combination $\sum \lambda_i Q_i$, and hence by I(b) there must exist a semi-definite $\sum \lambda_i Q_i$.

5. The system of equations $(Q_i \cdot B) = 0$. The substance of Theorem I can be equivalently expressed in terms of the system of equations

$$(5) \qquad (Q_i \cdot B) = 0, \qquad i = 1, 2, \cdots, m,$$

where the Q_i are given quadratic forms and B is a quadratic form restricted by the system of equations.

THEOREM II. *The system of equations* (5)

(a) *can admit only indefinite solutions B if the Q_i admit a definite linear combination,*

(b) *admits a definite solution B if every linear combination of the Q_i is indefinite,*

(c) *admits a semi-definite (but no definite) solution B if the Q_i admit a semi-definite (but no definite) linear combination.*

It will be noted that the system certainly admits some solution B except in the case where there exists a definite linear combination of the Q_i. In this latter case there may be indefinite solutions or *no* solution as illustrated in the following two examples.

EXAMPLE 1. The two forms

$$Q_1 = 2x_1^2 - x_2^2, \qquad Q_2 = x_1^2 - 2x_2^2$$

admit the definite linear combination $Q_1 - Q_2 = x_1^2 + x_2^2$, and the corresponding system (5) admits the indefinite solution $B = x_1 x_2$.

EXAMPLE 2. The three forms

$$Q_1 = 2x_1^2 - x_2^2, \qquad Q_2 = x_1^2 - 2x_2^2, \qquad Q_3 = x_1 x_2$$

admit the definite linear combination $Q_1 - Q_2 - Q_3 = x_1^2 - x_1 x_2 + x_2^2$, but the corresponding system (5) admits no solution form B.

CARNEGIE INSTITUTE OF TECHNOLOGY

NOTE ON A CONJECTURE DUE TO EULER

E. T. BELL

Euler's conjecture (1772) that

$$x_1^n + \cdots + x_t^n = x^n,$$

where n is an integer greater than 3 and $2 < t < n$, has no solution in rational numbers $x_1, \cdots, x_t, x$ all different from zero, is still unsettled even in its first case, $n = 4$, $t = 3$. It may therefore be of some interest to note a solution of this equation for any $n > 3$ and any $t > 1$ in terms of (irrational) algebraic numbers, which can be made algebraic integers by suitable choice of a homogeneity parameter, all different from zero, all the numbers being polynomials in numbers of degree $2d$, where $4d \leq 2n - 5 + (-1)^n$. If solutions differing only by a parameter are not considered distinct, there are at least d^{t-1} sets of solutions $x_1, \cdots, x_t, x$.

The solutions described are

$$x_1 = u, \qquad x_2 = r_{t-1} u, \qquad x = (1 + r_1) \cdots (1 + r_{t-1}) u;$$

$$x_j = r_{t-j+1}(1 + r_{t-j+2})(1 + r_{t-j+3}) \cdots (1 + r_{t-1}) u, \qquad j = 3, \cdots, t,$$

where u is a parameter and the r's are any roots, the same or different, of any factor $F_n(r)$, irreducible in the field of rational numbers, of

$$f(r) \equiv \sum_{s=1}^{n-1} (n, s) r^{n-s-1},$$