SOME PROPERTIES OF MEASURABLE FUNCTIONS

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1. **Introduction.** Throughout this paper the letter I will denote some fixed closed interval and f a numerically valued measurable function on I. It is our purpose to establish certain general properties of f. We point out in §4 that two theorems of Banach are almost immediate consequences of these properties. We suspect that further use can be made of our results.

2. **Some notations.** We define

$$X^{\wedge} = \underset{y}{E}\,[y = f(x) \text{ for some } x \in X], \qquad X \subset I,$$

$$Y^{\vee} = \underset{x}{E}\,[f(x) \in Y], \qquad Y \subset I^{\wedge}.$$

Writing $X^{\wedge\vee} = (X^{\wedge})^{\vee}$ and $Y^{\vee\wedge} = (Y^{\vee})^{\wedge}$ we note that the relations

$$X \subset X^{\wedge\vee}, \qquad Y = Y^{\vee\wedge},$$

$$\left(\sum_{n=1}^{\infty} X_n\right)^{\wedge} = \sum_{n=1}^{\infty} X_n^{\wedge}, \qquad \left(\sum_{n=1}^{\infty} Y_n\right)^{\vee} = \sum_{n=1}^{\infty} Y_n^{\vee},$$

$$\left(\prod_{n=1}^{\infty} X_n\right)^{\wedge} \subset \prod_{n=1}^{\infty} X_n^{\wedge}, \qquad \left(\prod_{n=1}^{\infty} Y_n\right)^{\vee} = \prod_{n=1}^{\infty} Y_n^{\vee},$$

$$X_1^{\wedge} - X_2^{\wedge} \subset (X_1 - X_2)^{\wedge}, \qquad Y_1^{\vee} - Y_2^{\vee} = (Y_1 - Y_2)^{\vee},$$

$$YX^{\wedge} = (Y^{\vee}X)^{\wedge},$$

hold whenever $X, X_1, X_2, \cdots$ are subsets of I and $Y, Y_1, Y_2, \cdots$ are subset of $I^{\wedge}$.

We further define

$$\{y\} = \underset{z}{E}\,[z = y],$$

$$\mathfrak{F} = \underset{y}{E}\,[\{y\}^{\vee} \text{ is finite}],$$

$$\mathfrak{K} = \underset{y}{E}\,[\{y\}^{\vee} \text{ has at least } \aleph_0 \text{ elements}],$$

$$\mathfrak{Q} = \underset{y}{E}\,[\{y\}^{\vee} \text{ has more than } \aleph_0 \text{ elements}].$$

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We denote the (outer) linear Lebesgue measure of a set A by $|A|$.

3. **Some theorems concerning** f.

THEOREM 3.1. *If A is a measurable subset of I, then there is a measurable subset B of A such that $B^{\wedge}=A^{\wedge}$ and f is univalent on B.*

PROOF. Let[1] $0=C_0\subset C_1\subset C_2\subset\cdots$ be closed subsets of A relative to each of which f is continuous and for which

$$C=\sum_{n=1}^{\infty}C_n, \qquad |A-C|=0.$$

For each positive integer n let[2] D_n be a measurable subset of C_n such that $D_n^{\wedge}=C_n^{\wedge}$ and f is univalent on D_n. Let

$$B_1=\sum_{n=1}^{\infty}[D_n-C_{n-1}^{\wedge\vee}].$$

Noting that $C_{n-1}^{\wedge}$ and $C_nC_{n-1}^{\wedge\vee}$ are closed and also that

$$D_n-C_{n-1}^{\wedge\vee}=D_n-C_nC_{n-1}^{\wedge\vee}, \qquad n=1,2,3,\cdots,$$

we see that B_1 is a measurable subset of C.

Now let $y_0\in C^{\wedge}$. Let k be the least integer in

$$\underset{n}{E}\,[y_0\in C_n^{\wedge}].$$

Since

$$y_0\in C_k^{\wedge}-C_{k-1}^{\wedge}=D_k^{\wedge}-C_{k-1}^{\wedge}\subset(D_k-C_{k-1}^{\wedge\vee})^{\wedge},$$

take a number x_0 such that $f(x_0)=y_0$ and

$$x_0\in[D_k-C_{k-1}^{\wedge\vee}]\subset B_1.$$

Thus $y_0\in B_1^{\wedge}$. Moreover if $x_0\neq x\in B_1$ with $f(x)=y_0$, we could, for some integer $l>k$, successively infer the relations

$$x\in D_l-C_{l-1}^{\wedge\vee}, \qquad y_0\in C_k^{\wedge}\subset C_{l-1}^{\wedge}. \qquad x\in C_{l-1}^{\wedge\vee},$$

the first and last of which contradict.

Consequently $C^{\wedge}\subset B_1^{\wedge}$, f is univalent on B_1, $C^{\wedge}=B_1^{\wedge}$.

Let $\alpha=A-C^{\wedge\vee}$ and select a set $B_2\subset\alpha$ such that $B_2^{\wedge}=\alpha^{\wedge}$ and f is univalent on B_2. Now

[1] See Saks, *Theory of the integral*, Warsaw, 1937, p. 72.

[2] See Saks, loc. cit. p. 282.

$$B_2 \subset \alpha \subset A - C, \qquad |B_2| = 0.$$

Defining $B = B_1 + B_2$ we see that B is measurable and that

$$\begin{aligned} A^{\wedge} \supset B^{\wedge} &= B_1^{\wedge} + B_2^{\wedge} \\ &= C^{\wedge} + (A - C^{\wedge\vee})^{\wedge} \supset C^{\wedge} + (A^{\wedge} - C^{\wedge}) = A^{\wedge}. \end{aligned}$$

Lastly f is univalent on B; for otherwise, in view of its univalence on B_1 and B_2, we could select points $x_1 \in B_1$ and $x_2 \in B_2$ with $f(x_1) = f(x_2)$ and deduce the false proposition

$$x_2 \in B_2 B_1^{\wedge\vee} \subset \alpha C^{\wedge\vee} = (A - C^{\wedge\vee}) C^{\wedge\vee} = 0.$$

The proof is complete.

LEMMA 3.2. *If f is continuous relative to each of the closed sets $A_1 \subset A_2 \subset A_3 \subset \cdots$ with*

$$A = \sum_{j=1}^{\infty} A_j \subset I,$$

then

$$S_n = A \cdot \left(\underset{y}{E} \left[A\{y\}^{\vee} \text{ has at least } n \text{ elements} \right] \right)^{\vee}$$

is a Borel set for each positive integer n.

PROOF. Let n be a positive integer.

For each positive integer j let W_j be that subset of Euclidean n-space such that $P = (P_1, P_2, \cdots, P_n)$ is in W_j if and only if

$$P_i \in A_j, \qquad i = 1, 2, \cdots, n,$$

$$P_i - P_{i-1} \geqq 1/j, \qquad i = 2, 3, \cdots, n.$$

For $j = 1, 2, \cdots 3, \cdots$ note that W_j is bounded and closed and let

$$B_j = A_j \underset{x}{E} \left[\text{there is a } P \in W_j \text{ with } f(x) = f(P_i) \text{ for } i = 1, 2, \cdots, n \right].$$

It follows that

$$S_n = \sum_{j=1}^{\infty} B_j. \tag{1}$$

Now let j_0 be a positive integer and $x_1, x_2, x_3, \cdots$ be points of B_{j0} such that

$$\lim_{m\to\infty} x_m = x_0 \in I.$$

Clearly $x_0 \in A_{j_0}$. There are points $P^1, P^2, P^3, \cdots$ of W_{j_0} such that

$$f(x_m) = f(\overset{m}{P_i}), \qquad m = 1, 2, 3, \cdots; i = 1, 2, 3, \cdots, n.$$

Let α be a set of integers and P^0 be a point of the compact set W_{j_0} such that

$$\lim_{m\to\infty,\, m\in\alpha} P^m = P^0.$$

Since f is continuous relative to A_{j_0} we conclude

$$f(\overset{0}{P_i}) = \lim_{m\to\infty,\, m\in\alpha} f(\overset{m}{P_i}) = \lim_{m\to\infty,\, m\in\alpha} f(x_m) = f(x_0),$$

$$i=1, 2, \cdots, n,$$

which implies $x_0 \in B_{j_0}$. Hence B_{j_0} is closed.

In view of (1) the proof is complete.

THEOREM 3.3. *There is a measurable*[3] *set C such that $C^\wedge = \mathfrak{R}$ and $(\{y\}^\vee - C)$ is finite for each $y \in I^\wedge$.*

PROOF. Retaining the notation of the statement of 3.2 and demanding in addition that $|I-A| = 0$, we define

$$S = A\cdot\left(\underset{y}{E}\,[A\{y\}^\vee \text{ is infinite}]\right)^\vee, \tag{2}$$

$$T = (I - A)\cdot\left(\underset{y}{E}\,[(I - A)\{y\}^\vee \text{ is infinite}]\right)^\vee, \tag{3}$$

$$C = S + T.$$

The set C is measurable because $S = \prod_{n=1}^\infty S_n$ and $|T| = 0$. The fact that $C^\wedge \subset \mathfrak{R}$ may be easily verified by deleting A and $(I-A)$ from (2) and (3), respectively.

The assumption that $(A\{y\}^\vee - C)$ is infinite leads to the relations

$$A\{y\}^\vee \subset S \subset C, \qquad A\{y\}^\vee - C = 0,$$

the second of which is contradictory. Similarly the assumption that $((I-A)\{y\}^\vee - C)$ is infinite leads to a contradiction.

Thus $\{y\}^\vee - C$ is finite for each $y \in I^\wedge$, $C\{y\}^\vee$ is infinite (and non-

[3] Using the fact that there is a perfect set of measure zero which is decomposable into a continuum of disjoint perfect sets, it can be shown that a judicious choice of our function f insures that neither $\mathfrak{R}^\vee$ nor $\mathfrak{Q}^\vee$ is measurable.

vacuous) for $y \in \mathfrak{R}$; consequently $\mathfrak{R} \subset C^\wedge$.

Hence $\mathfrak{R} = C^\wedge$.

THEOREM 3.4.[4] *Let* $\epsilon > 0$. *Then*

(i) *there is a measurable set* $L \subset I$ *with* $L^\wedge = \mathfrak{R}$ *and* $|L| < \epsilon$;

(ii) *there is a measurable set* $R \subset I$ *with* $R^\wedge = \mathfrak{Q}$ *and* $|R| = 0$.

PROOF. Let C be a measurable set such that $C^\wedge = \mathfrak{R}$ and $\{y\}^\vee - C$ is finite for each $y \in \mathfrak{R}$.

Let F be the family of measurable sets $X \subset C$ such that $X\{y\}^\vee$ is finite for each $y \in X^\wedge$. Note that $0 \in F \neq 0$ and define

$$M = \sup_{X \in F} |X|$$

and let $X_1, X_2, X_3, \cdots$ be sets in F such that $\lim_{n\to\infty}|X_n| = M$. Letting

$$S_n = \sum_{j=1}^{n} X_j, \qquad n = 1, 2, 3, \cdots,$$

observe that $X_n \subset S_n \in F$. Thus $|X_n| \leqq |S_n| \leqq M$ for $n = 1, 2, 3, \cdots$ and consequently

$$|S| = \lim_{n\to\infty} |S_n| = M \qquad \text{where } S = \sum_{n=1}^{\infty} S_n.$$

Let n_0 be an integer such that $|S - S_{n_0}| < \epsilon$. Let T be a measurable (3.1) subset of $C - S$ such that $T^\wedge = (C-S)^\wedge$ and f is univalent on T.

Defining

$$L = T + (S - S_{n_0}), \quad R = T\mathfrak{Q}^\vee,$$

we verify first that $|T| = 0$. If $|T|$ were greater than 0 there would be an integer k such that

$$|S_k| > M - |T|, \qquad (T + S_k) \in F, \qquad TS_k \subset TS = 0,$$

$$M \geqq |T + S_k| = |T| + |S_k| > M, \qquad M > M.$$

Consequently L and R are measurable, $|L| < \epsilon$, $|R| = 0$.

Next since $S_{n_0}\{y\}^\vee$ is at most finite, $S\{y\}^\vee$ is at most countable and $C\{y\}^\vee$ has the same power as $\{y\}^\vee$ for $y \in \mathfrak{R}$, we see that

$$\mathfrak{R} \subset (C - S_{n_0})^\wedge \subset \mathfrak{R}, \qquad \mathfrak{Q} \subset (C - S)^\wedge = T^\wedge.$$

From these relations it follows almost immediately that $L^\wedge = \mathfrak{R}$ and $R^\wedge = \mathfrak{Q}$.

[4] It is evident from Theorem 3.1 that Theorem 3.4 remains true if we add the requirement that f be univalent on L and on R.

4. **Some applications.** We say our function f satisfies condition S if and only if for each $\epsilon>0$ there is a $\delta>0$ such that

$$|X^\wedge\cdot[-1/\epsilon, 1/\epsilon]| < \epsilon \quad \text{whenever } X \subset I,\ |X| < \delta;$$

f satisfies condition N if and only if

$$|X^\wedge| = 0 \quad \text{whenever } X \subset I,\ |X| = 0;$$

f satisfies condition T_1 if and only if $|\mathfrak{K}| = 0$; and f satisfies condition T_2 if and only if $|\mathfrak{Q}| = 0$.

The following two theorems of Banach[5] are essentially corollaries of 3.4.

THEOREM 4.1. *If f satisfies condition N, then it satisfies T_2.*

THEOREM 4.2. *A necessary and sufficient condition that f satisfy condition S is that it satisfy both N and T_1.*

Theorem 4.1 and the *necessity* in 4.2 are immediate consequences of 3.4. The *sufficiency* in 4.2 may be proved as follows:

Suppose f satisfies N and T_1 but not S. Then we can find a number $\epsilon>0$ and measurable subsets $A_1 \supset A_2 \supset A_3 \supset \cdots$ of I such that

$$\lim_{n\to\infty} |A_n| = 0, \qquad |A_n^\wedge| \geqq \epsilon, \qquad A_n^\wedge \subset [-1/\epsilon, 1/\epsilon], \qquad n = 1, 2, 3, \cdots.$$

We note[6] that $A_n^\wedge$, $I^\wedge$, $\mathfrak{K}$, $\mathfrak{F}$, are measurable and define

$$A = \prod_{n=1}^{\infty} A_n.$$

Since the product of a descending sequence of nonvacuous *finite* sets is nonvacuous, we have

$$\mathfrak{F}\prod_{n=1}^{\infty} A_n^\wedge \subset \left(\prod_{n=1}^{\infty} A_n\right)^\wedge = A^\wedge.$$

Hence

$$|A^\wedge| \geqq \left|\mathfrak{F}\prod_{n=1}^{\infty} A_n^\wedge\right| = \lim_{n\to\infty} |\mathfrak{F}A_n^\wedge| = \lim_{n\to\infty} |A_n^\wedge| \geqq \epsilon$$

in contradiction to the relations

$$|A| = \lim_{n\to\infty} |A_n| = 0, \qquad |A^\wedge| = 0.$$

[5] See Saks, loc. cit. p. 284.

[6] If f satisfies N, then $X^\wedge$ is measurable, whenever X is a measurable subset of I.

Theorem 4.2 is proved.

5. **Generalizations.** Suppose E is a metric space and ϕ is a measure over[7] E such that: closed sets are ϕ measurable in the sense of Carathéodory;[8] every ϕ measurable set is the sum of an F_σ and a set of ϕ measure zero; E is a countable sum of compact sets; $\phi(E) < \infty$.

Let H be another metric space. We say a function q is ϕ finitely valued, if it is on E, its range is a finite subset of H and

$$\underset{x}{E}\,[q(x) = y]$$

is ϕ measurable for each $y \in H$. We call a function ϕ measurable, if it is ϕ almost everywhere in E the limit of a sequence of ϕ finitely valued functions.

To generalize our results we replace I by E, Lebesgue measure by ϕ, f by a ϕ measurable function g, the word "closed" in the statement of 3.2 by "compact." Conditions S, N, T_1, T_2 are to be interpreted in terms of such a measure ψ over H that closed subsets of H are ψ measurable and bounded sets have finite ψ measure.

All our theorems remain true under these conditions with properly adjusted notation. Leaving details to the reader, we solve the only non-trivial problem arising in this extension by the following:

THEOREM 5.1. *If g is continuous relative to the compact set $C \subset E$, then there is a Borel set $B \subset C$ such that $B^\wedge = C^\wedge$ and g is univalent on B.*

PROOF. Select[9] a continuous function h whose domain is a perfect set $A \subset [0, 1]$ and whose range is C.

For $n = 1, 2, 3, \cdots$, let A_n be the set such that $t \in A_n$ if and only if $t \in A$ and the relations

$$s \in A, \quad s \leqq t - n^{-1}, \quad g[h(s)] = g[h(t)],$$

are incompatible. Note that A_n is open with respect to A. Let

$$B_n = C \cdot \underset{x}{E}\,[x = h(t) \text{ for some } t \in A_n],$$

$$B = \prod_{n=1}^{\infty} B_n.$$

Observe that B is an $F_{\sigma\delta}$.

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[7] We say Λ is over E if and only if Λ is a function whose domain is the set of subsets of E.

[8] See H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, p. 424.

[9] See W. Sierpinski, *Introduction to general topology*, Toronto, 1934, p. 166.

Now choose $y_0 \in C^\wedge$. Taking

$$t_0 = \inf A \cdot \underset{t}{E} \, [g[h(t)] = y_0], \qquad x_0 = h(t_0),$$

we easily check that

$$t_0 \in \prod_{n=1}^{\infty} A_n, \qquad x_0 \in B, \qquad g(x_0) = y_0.$$

If $x_0 \neq x_1 \in B$ with $g(x_1) = y_0$, we infer that

$$T = A \cdot \underset{t}{E} \, [h(t) = x_1]$$

is a closed set lying entirely to the right of t_0 with distance $d > 0$ from t_0; hence $n > d^{-1}$ implies $T \subset A - A_n$ which in turn implies $x_1 \in E - B_n \subset E - B$ contrary to the assumption $x_1 \in B$.

Theorem 5.1 is proved.

Now we replace the condition that $\phi(E) < \infty$ by the hypothesis:

$$E = \sum_{n=1}^{\infty} E_n, \qquad E_n E_m = 0 \quad \text{for} \quad n \neq m,$$

$$E_n \text{ is } \phi \text{ measurable with } \phi(E_n) < \infty.$$

Obviously Theorem 3.4 part (i) and Theorem 4.2 do not[10] hold under these new conditions. However 3.1, 3.2, 3.3, 3.4 part (ii), 4.1 are still valid. To prove this it is sufficient to know of the existence of a measure Φ over E such that: $\Phi(E) < \infty$; a set is Φ measurable if and only if it is ϕ measurable; a set has Φ measure zero if and only if it has ϕ measure zero.

Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ be *positive* numbers such that

$$\sum_{n=1}^{\infty} \lambda_n \phi(E_n) < \infty$$

and define Φ over E by the relation:

$$\Phi(X) = \sum_{n=1}^{\infty} \lambda_n \phi(X E_n) \qquad \text{for } X \subset E.$$

The reader will find no difficulty in checking that Φ is a measure with the required properties.

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[10] Counterexample: $g(x) = \sin x, \; -\infty < x < +\infty$.