SECTIONS OF CONTINUOUS COLLECTIONS

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In the present note we establish the following

THEOREM. *Suppose G is a continuous collection*[1] *of closed and compact sets filling a separable metric space X. Suppose further that the space G, considered as a decomposition space, has dimension at most n. Then there is a closed subset K of X, such that for each $g \in G$, the set $g \cdot K$ is nonvacuous and consists of at most $(n+1)$ points.*

We call such a point set K an $(n+1)$-section of the collection G. Thus a 1-section of G is a true section. G. T. Whyburn[2] has shown that if the elements of G are 0-dimensional and G is a dendrite, then G admits a true section. The present result gives only a 2-section, but there is no hypothesis on the dimension of the elements of G. For $n=1$, it is known that in general G does not admit a true section. For $n>1$ it is not known whether the present result gives the best possible constant.

We first establish the theorem in the 0-dimensional case.

LEMMA. *Suppose G is 0-dimensional, and ϵ is a given positive number. Suppose W is an open set in X such that $W \cdot g \neq 0$ for each $g \in G$. Then there is an open set E in X such that $\overline{E} \subset W$, $E \cdot g \neq 0$ for every $g \in G$, and the diameter of $E \cdot g < \epsilon$ for each $g \in G$.*

Let $f(x)$ be a homeomorphism of M, a subset of the Cantor set, into G.[3] In the product space $M \times X$, consider the set A of points (x, y) with $x \in M$ and $y \in f(x)$. For $x \in X$ there is a unique $y = y(x)$ in M such that $x \in f(y)$. The function $t(x) = (y(x), x)$ is a homeomorphism of X into A.

In the space A, the open set $t(x)$ and the continuous collection H of elements $t(g)$ for $g \in G$ satisfy the properties of W and G stated in the hypothesis of the lemma. Furthermore, the diameter of a set Z in A is not smaller than the diameter of $t^{-1}(Z)$. Hence all we need show is that there exists an open set E satisfying the theorem relative to the open set $t(W) = U$ and the continuous collection H.

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[1] *A continuous collection filling a space X*, is a collection G of sets g such that: (1) If $x \in X$, then $x \in g$ for exactly one g. (2) If $x \in g$, $x_n \in g_n$ and $x_n \rightarrow x$, then $\lim g_n = g$.

[2] *A theorem on interior transformations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 414–416.

[3] P. Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math. vol. 7 (1926) p. 77.

For each $p \in U$ there is an open set U_p such that (1) $U_p \supset p$, (2) $\overline{U}_p \subset U$, (3) the diameter of $U_p < \epsilon$, and (4) the projection[4] V_p of U_p upon M is both open and closed. The collection $\{V_p\}$ is an open covering of M and therefore there is a countable subcollection $\{V_{p_i}\}$ covering M. The collection of sets $\{W_i\}$ where W_i is defined by the relations

$$W_1 = V_{p_1}, \qquad W_i = V_{p_i} - \sum_{j=1}^{i-1} V_{p_j}$$

is a covering of M by mutually exclusive open sets. Let Y_i denote the open subset of U_{p_i} whose projection is W_i and let $E = \sum_{i=1}^{\infty} Y_i$. The open set E has the required properties relative to the space A, the open set U, and the continuous collection H.

In order to prove that $\overline{E} \subset U$ it is sufficient to show that $\overline{E} = \sum_{i=1}^{\infty} \overline{Y}_i$. Suppose $p \in \overline{E}$ and $p \notin \sum_{i=1}^{\infty} \overline{Y}_i$. Then there is a sequence $p_n \rightarrow p$ and $p_n \in Y_{i_n}$. Suppose $p \in g$, and $\pi(g) \subset W_j$, where π denotes the projection of A on M. Since $p_n \rightarrow p$, $\pi(p_n) \rightarrow \pi(p)$. But $\pi(p_n) \not\subset W_j$ for more than a finite number of n. This contradicts the fact that W_j is open.

Now E intersects each g since the sequence $\{W_i\}$ is a covering of M. Also, since the sets W_i are mutually exclusive, if $Y_i \cdot g \neq 0$ then $Y_j \cdot g = 0$ for $i \neq j$. Then, as $Y_i \subset U_{p_i}$ and the diameter of $U_{p_i} < \epsilon$, the diameter of $E \cdot g < \epsilon$ for every $g \in H$. This proves the lemma.

The theorem for the 0-dimensional case follows by considering a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and a sequence of open sets $\{E_n\}$ such that $\overline{E}_{n+1} \subset E_n$, $E_n \cdot g \neq 0$ for $g \in G$, and the diameter of $E_n \cdot g < \epsilon_n$, for $g \in G$. The common part K of the sets E_n is closed. For $g \in G$, the set $K \cdot g$ consists of exactly one point, since g is compact and $\epsilon_n \rightarrow 0$.

The theorem for the n-dimensional case follows by considering an at most $(n+1)$-to-one closed mapping $f(x)$ of a subset M of the Cantor set[5] into G. In the product space $M \times X$ consider the set A of points (y, x) with $y \in M$ and $x \in f(y)$. The sets (y, x) for y fixed and $x \in f(y)$ form a 0-dimensional continuous collection H which fills A. The mapping $t(y, x) = x$ is a closed, at most $(n+1)$-to-one mapping of A into X. By the theorem for the 0-dimensional case, there is a true section K of the collection H in the space A. The set $t(K)$ gives the required $(n+1)$-section of the continuous collection G.

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[4] That is, V_p is the set of $x \in M$ such that $(x, y) \in U_p$.

[5] See J. H. Roberts, *A theorem on dimension*, Duke Math. J. vol. 8 (1941) p. 572, Theorem 9.1. The mapping ϕ_n as actually defined is a closed mapping, although this result is not specifically stated in the theorem.