problems associated with the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0$$

for various types of boundary conditions when the boundary is rectangular.

PURDUE UNIVERSITY

ON THE CONVERGENCE OF A CONTINUED FRACTION

T. F. GLASS AND WALTER LEIGHTON

It is known [1] that sufficient conditions for the convergence of the continued fraction

$$b_0 + \frac{a_1}{1+}\,\frac{a_2}{1+}\cdots, \tag{1}$$

where the elements are complex numbers, are

$$|a_2| \geqq 5, \quad |a_{2n}| \geqq 25/4, \quad |a_{2n-1}| \leqq 1/4, \quad n = 2, 3, 4, \cdots. \tag{2}$$

The purpose of this note is to extend this result.

THEOREM. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ $(n=1, 2, 3, \cdots)$ *and if the numbers* $a_{2n} = \rho_{2n}e^{i\theta_{2n}}$ $(n=1, 2, 3, \cdots)$ *satisfy the conditions*

$$\rho_{2n} \geqq 2(1+r)^2[1-\cos(\theta_{2n}+\theta_0)], \qquad 0 \leqq \theta_{2n} < \pi - \theta_0, \tag{3}$$

$$\rho_{2n} \geqq 4(1+r)^2, \qquad \pi - \theta_0 \leqq \theta_{2n} \leqq \pi + \theta_0, \tag{4}$$

$$\rho_{2n} \geqq 2(1+r)^2[1-\cos(\theta_{2n}-\theta_0)], \qquad \pi + \theta_0 < \theta_{2n} \leqq 2\pi, \tag{5}$$

where $\theta_0 = 2$ arc sin r, *the continued fraction* (1) *converges.*

To prove the theorem we employ the continued fraction

$$1 + \frac{x_1}{1+}\,\frac{x_2}{1+}\cdots \tag{6}$$

where

$$x_{2n} = \frac{(1+a_{2n-1})(1+a_{2n+1})}{a_{2n}}, \qquad n = 2, 3, \cdots, \tag{7.1}$$

Presented to the Society, September 5, 1941; received by the editors May 7, 1942.

$$(7.2)\qquad x_{2n+1} = a_{2n+1}, \qquad n = 1, 2, \cdots,$$

$$(7.3)\qquad x_1 = -a_1, \qquad x_2 = \frac{1 + a_3}{a_2}.$$

Since the convergence of (1) is independent of the choice of b_0, we set $b_0 = 1 - a_1$, and it follows [1] that the $2n$th convergent of (1) formally equals the $(2n+1)$st convergent of (6), while the $(2n+1)$st convergent of (1) is equal to the $2n$th convergent of (6). Thus (1) and (6) will converge or diverge together. By the first hypothesis of the theorem and (7.2) the numbers x_{2n+1} are bounded and lie in $\overline{R}$ the closed parabola $|z| - \mathfrak{R}(z) = 1/2$. It is thus sufficient to prove that the numbers x_{2n} defined in (7.1) and (7.3) subject to conditions (3), (4), and (5) are bounded and lie in the parabolic region described above [2].

To this end let

$$\begin{aligned} s_{2n+1} &= (1 + a_{2n-1})(1 + a_{2n+1}) = a_{2n}x_{2n} \\ &= r_{2n+1}e^{i\phi_{2n+1}}, \qquad n = 2, 3, \cdots, \end{aligned}$$

and set $x_{2n} = t_{2n}e^{i\omega_{2n}}$. It is clear that $t_{2n} \leqq \max r_{2n+1}/\min \rho_{2n} = (1+r)^2/\min \rho_{2n}$.

First suppose that a_{2n} lies in the region defined by (4). Then $\min \rho_{2n} = 4(1+r)^2$, $t_{2n} \leqq 1/4$ and x_{2n} will lie in $\overline{R}$. Next suppose that (3) holds, that is, that a_{2n} lies outside the cardioid $\rho = 2(1+r)^2[1 - \cos(\theta + \theta_0)]$ and in the angle $0 \leqq \theta < \pi - \theta_0$. Hence $\min \rho_{2n} = 2(1+r)^2[1 - \cos(\theta+\theta_0)]$ for each θ and thus

$$(8)\qquad t_{2n} \leqq \frac{1}{2[1 - \cos(\theta_{2n} + \theta_0)]}, \quad 0 \leqq \theta_{2n} < \pi - \theta_0.$$

Further $\omega_{2n} = \phi_{2n+1} - \theta_{2n}$ from which it follows that $-\theta_0 - \theta_{2n} \leqq \omega_{2n} \leqq \theta_0 - \theta_{2n}$, and hence that $-\pi \leqq \omega_{2n} \leqq \theta_0$. It is clear that the right-hand member of (8) decreases steadily from $1/2(1 - \cos\theta_0)$ to $1/4$ as θ_{2n} increases from 0 to $\pi - \theta_0$ and at the same time ω_{2n} decreases steadily from ϕ_{2n+1} to $\phi_{2n+1} + \theta_0 - \pi$. The proof for the case when (3) holds may now be completed by proving that the point $(1/2\ (1 - \cos\theta_0), \theta_0)$ and the points

$$(9)\qquad (1/2[1 - \cos(\theta + \theta_0)], -\theta - \theta_0), \qquad 0 \leqq \theta < \pi - \theta_0,$$

lie in $\overline{R}$, as an examination of a simple figure will show. The first point evidently lies on the parabola since the equation of the parabola in polar coordinates is

$$(10)\qquad r = \frac{1}{2(1 - \cos\theta)}.$$

The substitution $\lambda=-\theta-\theta_0$ proves immediately that the points (9) lie on (10).

By symmetry it is clear that when a_{2n} lies in the region defined by (5) the corresponding complex number x_{2n} lies in $\overline{R}$. This completes the proof of the theorem.

COROLLARY 1. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ *and* $|a_{2n}| \geqq 4(1+r)^2$, *the continued fraction* (1) *converges.*

It is clear that an analogous theorem to the above may be proved with the roles of the even and the odd elements interchanged.

BIBLIOGRAPHY

1. Walter Leighton and H. S. Wall, *On the transformation and convergence of continued fractions*, Amer. J. Math. vol. 58 (1936) pp. 267–281.

2. W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 155–172.

THE MARION INSTITUTE AND
THE RICE INSTITUTE