ON THE JOIN OF TWO COMPLEXES

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1. **Introduction.** In this note we point out an isomorphism between the $(r+1)$-dimensional Betti group of the join (defined below) of two complexes and a subgroup of the r-dimensional Betti group of the product of the two complexes. Using this isomorphism the Betti groups of the join are derived from those of the product in case the complexes are finite.[1]

2. **Definition of the join (K_1, K_2) of K_1 and K_2.** To define the join of two complexes we first define the join (σ, τ) of a p-dimensional simplex σ and a q-dimensional simplex τ, $p, q=0, 1, \cdots$. This join is a $(p+q+1)$-dimensional simplex with a p-dimensional side associated with σ and the opposite side, which is q-dimensional, associated with τ. These sides will not be distinguished from σ and τ, respectively. Now consider the complexes K_1 and K_2. Consider the set consisting of the simplexes σ_α of K_1, the simplexes τ_β of K_2, and the simplexes $(\sigma_\alpha, \tau_\beta)$. In a natural way this set forms a complex. We define the *join* (K_1, K_2) of K_1 and K_2 to be the first barycentric subdivision of this complex.

3. **The rays.** By the rays of (σ, τ) we mean the straight line segments each of which joins a point of σ and a point of τ. These rays cover (σ, τ). Also no two rays intersect except possibly at an end point. The rays of all $(\sigma_\alpha, \tau_\beta)$ of (K_1, K_2) are called the *rays*.

Let N_i, $i=1, 2$, be the subcomplex made up of the simplexes of (K_1, K_2) that have at least one vertex in K_i together with the faces of all such simplexes. It is known that each ray meets the intersection $N_1 \cap N_2$ in exactly one point.[2] Furthermore N_i and $N_1 \cap N_2$ can be homotopically deformed in N_i along the rays into K_i, $i=1, 2$.[2] It follows that $N_1 \cap N_2$ and the product $K_1 \times K_2$ are homeomorphic (the complexes being considered as point sets).

4. **The theorem.** We prove this theorem.

THEOREM 1. *There is an isomorphism between the $(r+1)$-dimensional*

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[1] The Betti groups of the join of two finite complexes are known. They were computed by H. Freudenthal in his paper *Die Bettischen Gruppen der Verbindung Zweier Polytope*, Fund. Math. vol. 29 (1937) pp. 145–150.

[2] For a proof see our paper *Simultaneous invariants of a complex and subcomplex*, Duke Math. J. vol. 5 (1939) pp. 62–71.

Betti group of (K_1, K_2) *and the subgroup of those homology classes of the r-dimensional Betti group of* $N_1 \cap N_2$ *which contain cycles that bound both in* N_1 *and* N_2, $r \geqq 0$; *all chains considered in this paper are finite and have integral coefficients.*

In this theorem K_1 and K_2 may be infinite.

PROOF. We note that $(K_1, K_2) = N_1 + N_2$. Hence we know that there is a homomorphism from the $(r+1)$-dimensional Betti group of $N_1 + N_2$ onto those homology classes of the r-dimensional Betti group of $N_1 \cap N_2$ which contain cycles that bound both in N_1 and N_2, $r \geqq 0$; furthermore, the kernel of this homomorphism consists of the homology classes that contain Summenzyklen, that is, cycles that are equal to the sum of two cycles, one in N_1 and the other[3] in N_2. To prove Theorem 1 we shall show that a Summenzyklus bounds in (K_1, K_2). To prove this, consider a cycle Z of N_1. This cycle can be homotopically deformed in N_1 along the rays into a singular cycle of K_1. We know that (K_1, K_2) contains the join of K_1 and an arbitrary point of K_2. Hence any cycle of K_1 can be homotopically deformed in (K_1, K_2) into a cycle of a vertex of K_2. Since the dimension of Z is greater than zero, it follows that Z is homologous to zero in (K_1, K_2). Similarly, a cycle of N_2 with dimension greater than zero bounds in (K_1, K_2).

5. **Some properties of** $N_1 \cap N_2$. From now on K_1 and K_2 are finite. For any chain A of a complex let $|A|$ denote the closure of the set of those simplexes at which[4] $A \neq 0$. We know that with each pair of chains $C_1 \subset K_1$ and $C_2 \subset K_2$ there is associated a chain $C_1 \times C_2 \subset N_1 \cap N_2$; the dimension of $C_1 \times C_2$ is the sum of the dimensions of C_1 and C_2; and $|C_1 \times C_2| = |C_1| \times |C_2|$.[5] It follows that the projection of $|C_1 \times C_2|$ into K_1 or K_2 along the rays is a subset of $|C_1|$ or $|C_2|$, respectively. Also if C_2 is a zero-dimensional cycle, and $C_2 = 0$ at all but one vertex of K_2 at which $C_2 = 1$, then $C_1 \times C_2$ and C_1 are isomorphic, and homotopic deformation of $C_1 \times C_2$ along the rays shows that $C_1 \times C_2 \sim C_1$ in N_1 (all the rays used in the deformation meet at a point of K_2).

Let

$$(1) \qquad Z_i^p \subset K_1$$

[3] See Alexandroff-Hopf, *Topologie*. I, p. 293, Theorem V. This theorem and its applicability were pointed out by the referee.

[4] An r-dimensional chain is a function defined over all r-dimensional simplexes of a complex.

[5] These and the following properties of $N_1 \cap N_2 = K_1 \times K_2$ are proved in Alexandroff-Hopf, *Topologie*. I, pp. 299–310.

and

(2) $$Z_j^q \subset K_2$$

be homology bases[6] for the p-dimensional cycles of K_1 and the q-dimensional cycles of K_2. We know that there is a set of s-dimensional chains $u_k^s \subset K_1$, $s=2, 3, \cdots$, in $(1=1)$ correspondence with the $(s-1)$-dimensional torsion coefficients of K_1, and there is a set of t-dimensional chains $v_l^t \subset K_2$, $t=2, 3, \cdots$, in $(1=1)$ correspondence with the $(t-1)$-dimensional torsion coefficients of K_2 such that a homology basis for the r-dimensional cycles of $N_1 \cap N_2$ is given by

(3) $$Z_{ij}^r = Z_i^p \times Z_j^q, \quad p + q = r, \text{ order of } Z_{ij}^r \neq 1,$$

and

(4) $$C_{kl}^r = c_{kl}^r(u_k^s \times v_l^t)^\cdot, \quad s + t = r + 1, c_{kl}^r \neq 1,$$

where c_{kl}^r is the reciprocal of the greatest common divisor of the torsion coefficients associated with u_k^s and v_l^t. Furthermore the order of Z_{ij}^r is the greatest common divisor of the orders[7] of Z_i^p and Z_j^q, and the order of C_{kl}^r is the reciprocal of c_{kl}^r.

6. **The cycles of $N_1 \cap N_2$ that bound both in N_1 and N_2.** We choose a cycle from (1) with $p=0$ and a cycle from (2) with $q=0$. Each of these cycles will be denoted by the same symbol Z_1^0.

THEOREM 2. *A homology basis for the subgroup mentioned in Theorem 1 consists of the following subset of* (3) *and* (4): *if* $r>0$, *the basis consists of* Z_{ij}^r *with both* $p>0$ *and* $q>0$, $Y_{ij}^r = Z_i^r \times (Z_1^0 - Z_j^0)$, $j \neq 1$, $X_{ij}^r = (Z_1^0 - Z_i^0) \times Z_j^r$, $i \neq 1$, *and all* C_{ij}^r; *if* $r=0$, *the basis consists of* $W_{ij}^0 = (Z_1^0 - Z_i^0) \times (Z_1^0 - Z_j^0)$, $i \neq 1, j \neq 1$.

PROOF. In the first place consider Z_{ij}^r with both $p \neq 0$ and $q \neq 0$. We know that Z_{ij}^r can be homotopically deformed in N_1 along the rays into $|Z_i^p|$, a p-dimensional subcomplex of K_1. Since $p<r$, this implies that Z_{ij}^r bounds in N_1. From symmetry, if $p \neq 0$ and $q \neq 0$, the cycle Z_{ij}^r bounds both in N_1 and N_2.

Consider next C_{kl}^r. Since $s+t=r+1$, $s>1$, and $t>1$, it follows that

[6] By a homology basis for the r-dimensional cycles of a complex we mean a set of cycles obtained by expressing the r-dimensional Betti group as the usual direct sum of free cyclic groups and finite cyclic groups whose orders are the torsion coefficients and by choosing a cycle from a generator of each summand.

[7] By the order of a cycle we mean the order of its homology class. Also it is understood that a free group has order zero and that the greatest common divisor of zero and a positive integer is that integer.

$s<r$ and $t<r$. Hence C^r_{kl} bounds both in N_1 and N_2 because C^r_{kl} is homotopic in N_1 to a cycle of $|u^s_k|$ and is homotopic in N_2 to a cycle of $|v^t_l|$.

Next consider Z^r_{ij} with $p=r\neq 0$. We can assume that $Z^0_j=0$ at all but one vertex of K_2 and that $Z^0_j=1$ at the exceptional vertex. Then as shown above $Z^r_{ij}\sim Z^r_i$ in N_1. From this we shall deduce that Z^r_{ij} does not bound in N_1. Suppose Z^r_{ij} does bound in N_1. Then $Z^r_i=\dot{F}$, $F\subset N_1$. We deform F in N_1 along the rays into the singular chain $F'\subset K_1$. The singular cycle $Z^r_i=\dot{F}'$ which contradicts the definition of Z^r_i.

From symmetry we see that Z^r_{ij} with $q=r\neq 0$ does not bound in N_2.

Now consider Y^r_{ij}. Since $Z^r_i\times Z^0_1$ and $Z^r_i\times Z^0_j$ are each homologous in N_1 to Z^r_i, it follows that Y^r_{ij} bounds in N_1. That Y^r_{ij} also bounds in N_2 follows from the fact that Y^r_{ij} can be homotopically deformed into a complex consisting of two vertices. From symmetry we know that X^r_{ij} bounds both in N_1 and N_2.

Finally if $r=0$, we see that W^0_{ij} bounds both in N_1 and N_2.

The theorem follows easily from these facts.

7. **The Betti groups of** (K_1, K_2). Theorems 1 and 2 imply this theorem.

FREUDENTHAL'S THEOREM.[1] *From* (1) *delete one cycle with* $p=0$ *and from* (2) *delete one cycle with* $q=0$; *each association of one of the remaining* Z^p_i *with one of the remaining* Z^q_j, $p+q=r$, *represents a generator of the* $(r+1)$*-dimensional Betti group of* (K_1, K_2); *the order of this generator is the greatest common divisor of the orders of* Z^p_i *and* Z^q_j; *furthermore each* C^r_{kl} *in* (4) *represents a generator of order* $1/c^r_{kl}$; *all such generators with their orders define the* $(r+1)$*-dimensional Betti group of* (K_1, K_2).

8. **The case** $r=-1$. The join (K_1, K_2) is connected because two points of K_i can be joined to an arbitrary point of K_j, $j\neq i$.

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