THE MODULAR SPACE DETERMINED BY A POSITIVE FUNCTION

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At the suggestion of T. H. Hildebrandt the authors undertook to determine the nature of the space of modular functions of E. H. Moore when the range $\mathfrak{P}$ is taken to be the infinite interval $-\infty < x < +\infty$ and the base matrix ϵ to be of the form

$$\epsilon(x, y) = \int_{-\infty}^{+\infty} e^{i(x-y)t} dV(t), \tag{1}$$

where V is a monotonically increasing bounded function. This form of ϵ is suggested by the work of Bochner on positive functions.[1] In this note we determine the form of functions modular as to ϵ and of the J-integral.

To avoid, at first, convergence questions we turn our attention to functions ϕ finite as to ϵ, that is, functions of the form

$$\phi(x) = \sum_{j=1}^{n} \epsilon(x, y_j)a_j = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV(t), \tag{2}$$

where

$$\lambda(t) = \sum_{j=1}^{n} a_j e^{-iy_j t}. \tag{3}$$

In the formulas (2) and (3) the a_j are arbitrary constants and the y_j are points on the interval $(-\infty, +\infty)$. It is known from standard results in the theory of modular and finite functions[2] that every function ϕ finite as to ϵ is modular and that

$$\begin{aligned} N\phi &= J\bar{\phi}\phi = \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k)a_k, \\ J\bar{\phi}_1\phi_2 &= [N(\phi_1 + \phi_2) - N(\phi_1 - \phi_2) - iN(\phi_1 + i\phi_2) \\ &\qquad + iN(\phi_1 - i\phi_2)]/4. \end{aligned} \tag{4}$$

Calculating the values of $N\phi$ and $J\bar{\phi}_1\phi_2$, we see that

$$J\bar{\phi}_1\phi_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV, \qquad N\phi = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

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[1] S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen, vol. 108 (1933), pp. 378–410.

[2] E. H. Moore, *General Analysis*, Part II, Philadelphia, 1939, pp. 94 ff.

To determine the form of an arbitrary modular function μ we consider a sequence

$$\phi_n(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda_n dV$$

of functions finite as to ϵ, converging mode 2 to a modular function[3] μ. Since ϕ_n converges strongly, it follows that

$$\lim_{m,n} N(\phi_m - \phi_n) = \lim_{m,n} \int_{-\infty}^{+\infty} | \lambda_m - \lambda_n |^2 dV = 0,$$

and hence there exists a measurable function λ such that λ^2 is integrable with respect to V and[4]

$$\lim_{n} \int_{-\infty}^{+\infty} | \lambda_n - \lambda |^2 dV = 0.$$

With the help of Schwarz' inequality one sees that

$$\mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t) dV, \qquad N\mu = \int_{-\infty}^{+\infty} | \lambda |^2 dV.$$

THEOREM 1. *To each modular function μ there corresponds a measurable function λ such that λ^2 is integrable with respect to V and*

$$(5) \qquad \mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t) dV(t) \quad \text{and} \quad N\mu = \int_{-\infty}^{+\infty} | \lambda |^2 dV.$$

Conversely, if λ is measurable and λ^2 integrable with respect to V, then the first of the formulas (5) *defines a modular function μ for which the second of these formulas is valid. If μ_1, μ_2 are two modular functions, then*

$$(6) \qquad J\bar{\mu}_1\mu_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV$$

where λ_1, λ_2 are the square integrable functions associated with μ_1, μ_2.

It remains to prove only the latter part of the theorem. To do this let $\xi(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda dV$, where λ is any measurable function such that λ^2 is integrable with respect to V, and let x_j, a_j ($j = 1, 2, \cdots, n$) be constants such that

[3] Ibid., p. 116.

[4] E. W. Hobson, *The Theory of Functions of a Real Variable*, 2d edition, 1926, vol. II, p. 246.

$$\sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV \leqq 1.$$

It then follows with the help of Schwarz' inequality that

$$\left| \sum_{j=1}^{n} \bar{a}_j \xi(x_j) \right|^2 = \left| \int_{-\infty}^{+\infty} \left(\sum_{j=1}^{n} \bar{a}_j e^{ix_j t} \right) \lambda dV \right|^2 \leqq \int_{-\infty}^{+\infty} |\lambda|^2 dV,$$

and hence ξ is modular.[5] The formula (6) follows at once from the second equation (5) and equation (4).

Finally, we seek conditions that the matrix ϵ should be proper. These are contained in the following result:

THEOREM 2. *The base matrix ϵ is proper if the measure function V is such that every set E whose complement has zero measure has a finite limit point.*

It is clear that

$$0 = \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV$$

implies the vanishing of the analytic function

$$\sum_{j=1}^{n} a_j e^{-ix_j t} \tag{7}$$

for almost all t. If the constants a_j were not all zero, the expression (7) would have a non-finite number of zeros in a bounded interval, which is false, and hence $a_1, \cdots, a_n$ are all zero and ϵ is a proper matrix.

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[5] Ibid., p. 84.