ON THE INTERIORITY OF REAL FUNCTIONS[1]

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If X and Y are metric spaces, a continuous transformation $f(X)=Y$ is said to be interior at a point x of X provided that if U is any open subset of X containing x, $f(x)$ is interior to $f(U)$ in Y. In 1928 Kuratowski[2] proved a theorem essentially to the effect that if X is compact, the set G of all points y in Y such that f is interior at all points of $f^{-1}(y)$ is a G_δ-set dense in Y. We first establish the following related result.

THEOREM. *If $f(x)$ is a real-valued continuous function defined on a locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$.*

PROOF. For each $y \varepsilon Y$ let Y_1 and Y_2 be the sets of all numbers in Y which are less than y and greater than y, respectively. Let $M_1(y)=f^{-1}(Y_1)$, $M_2(y)=f^{-1}(Y_2)$. Now there must exist[3] a countable subset C of Y such that if y is any point of $Y-C$,

$$f^{-1}(y) \subset \overline{M_1(y)} \cdot \overline{M_2(y)}.$$

For if not, there would exist an uncountable subset K of Y such that for any $y \varepsilon K$ there is a point $p_y \varepsilon f^{-1}(y)$ which fails to be a limit point either of $M_1(y)$ or of $M_2(y)$. Clearly we may suppose that for an uncountable subset of K_1 of K, $p_y \cdot \overline{M_1(y)}=0$. Now since M is separable and metric, there exists a countable sequence $R_1, R_2, \cdots$ of open sets in M such that if U is any open set in M and $p \varepsilon U$, there is an m such that $p \varepsilon R_m \subset U$. Hence, for each $y \varepsilon K_1$ there exists an integer m_y such that $p_y \subset R_{m_y}$ and $R_{m_y} \cdot M_1(y)=0$. But if y, $y' \varepsilon K_1$, $y'<y$, we have $p_{y'} \subset M_1(y)$ so that $m_y \neq m_{y'}$. This is absurd, since K_1 was uncountable. Hence we have established the existence of the set C as asserted above.

We shall show that f is interior at any point x of $M-f^{-1}(C)$. To this end let U be any open set in M containing x and let $y=f(x)$. Since M is locally connected, there exists a connected open subset V

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[1] This paper, in slightly different form, was sent to the editors of Fundamenta Mathematicae in June, 1939.

[2] See Fundamentae Mathematicae, vol. 11 (1926), p. 176.

[3] This could be established also with the aid of a lemma of Zarankiewicz. See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

of M with $x\subset V\subset U$. Now V contains points x_1 and x_2 of $M_1(y)$ and $M_2(y)$, respectively. Since $f(V)$ is connected and contains $f(x_1)$ and $f(x_2)$, it therefore contains the interval $[f(x_1), f(x_2)]$. Hence $f(x)$, which is interior to this interval, is interior to $f(V)$ and hence surely to $f(U)$.

Examples are easily constructed to show that our conclusion is invalid in the absence of either the local connectedness condition on M or the linearity restriction on the image space.

However, we can extend the theorem considerably. First we note the following corollary. (Note: All sets referred to in this paper are assumed separable and metric.)

COROLLARY. *Let $f(M)=Y=\sum_{i=1}^{\infty}L_i$ be continuous where M is locally connected, each L_i is open in Y and is homeomorphic with a subset of a line and $L_i\cdot L_j=0$ for $i\neq j$. There exists a countable subset C of L such that f is interior at every point of $M-f^{-1}(C)$.*

For under the conditions on the L_i it results at once that Y is homeomorphic with a subset of a line and hence with a set of real numbers. (For if L_i' is a subset of the open interval $(i-1, i)$ homeomorphic with L_i, $i=1, 2, \cdots$, clearly Y is homeomorphic with $\sum L_i'$.) Thus f is effectively a real continuous function on M and the corollary follows from the preceding theorem.

THEOREM. *Let $f(M)=K$ be continuous where M is locally connected and let D be the closure of the set of all points of order greater than 2 of K. There exists a countable subset C of $K-D$ such that f is interior at every point of $M-f^{-1}(D+C)$.*

PROOF. By continuity of f, $M-f^{-1}(D)=M'$ is open in M and hence is locally connected. Consider the mapping $f(M')=K-D=K'$. Since every point of K' is of order at most 2, K' is locally connected. Hence the components $R_1, R_2, \cdots$ of K' are open (thus countable). Further, each R_i is homeomorphic with a subset of a line, since it is a connected set all of whose points are of order at most 2. Thus by the above corollary there exists a countable subset C of K' such that f is interior at all points of $M'-f^{-1}(C)$. Obviously this yields our theorem.

COROLLARY. *If D is countable, there exists a countable subset C of K such that f is interior at all points of $M-f^{-1}(C)$.*

COROLLARY. *Any continuous mapping $f(M)=K$ of a locally connected set M into a graph K is interior at all points of $M-f^{-1}(C)$, where C is some countable subset of K.*

If X and Y are topological spaces, X_1 is a subset of X and $f(X_1) \subset Y$ and $\phi(X) \subset Y$ are single-valued continuous transformations, $\phi(x)$ is said to be an extension of f to X provided $f(X) = \phi(x)$ for $x \varepsilon X_1$. Extension theorems in various forms for continuous transformations are well known.[4] However, not much consideration seems to have been given to the important problem of carrying over to an extension ϕ of f properties which f may enjoy in addition to continuity. In this note a beginning in this direction is made with the following theorem.

THEOREM. *Let M be a subcontinuum of a cyclic locally connected compact continuum L and let $f(M) = (0,1)$ be continuous and such that for any y, $0 \leqq y \leqq 1$, $L - f^{-1}(y)$ is connected. There exists an extension $\phi(L) = (0, 1)$ of f to L which is interior at every point of $L - M$ and at every point of M where f is interior.*[5] *(Thus if f is interior on M, ϕ is interior on L.)*

PROOF. Let us decompose L upper semi-continuously[6] into the subsets $[f^{-1}(y)]$, $0 \leqq y \leqq 1$, of M and individual points of $L - M$. Call L' the hyperspace of this decomposition and let $h(L) = L'$ be the associated transformation. Clearly $h(M)$ is a simple arc $a'b'$ joining $a' = hf^{-1}(0)$ and $b' = hf^{-1}(1)$. By a theorem of the author,[7] there exists a non-alternating interior retracting transformation $g(x)$ retracting the cyclic chain $C(a,'b')$ in L' into $a'b'$. But since no one of the sets $f^{-1}(y)$ disconnects L and L is cyclic, it follows that L' is cyclic and hence $L' = C(a', b')$. Thus $g(x)$ retracts L' into $a'b'$.

Let us define

$$\phi(x) = fh^{-1}gh(x), \qquad x \varepsilon L.$$

Then ϕ has the required properties. For if $x \varepsilon M$, we have $h(x) \varepsilon a'b'$ so that $gh(x) = h(x)$; whence $h^{-1}gh(x) = h^{-1}h(x) = f^{-1}f(x)$ so that $fh^{-1}gh(x) = f(x)$. This proves $\phi(x) \equiv f(x)$ on M. To show that ϕ is interior at any point x where f is interior or at any point of $L - M$, let $x \varepsilon U$ where U is open in L. If $x \varepsilon M$, then since f is interior at x, $h[U \cdot M]$ contains an open subset V of $a'b'$ about $h(x)$. Hence $g(V) = V$

[4] See, for example, F. Hausdorff, Fundamenta Mathematicae, vol. 30 (1938), p. 40, and C. Kuratowski, ibid., p. 48.

[5] It is supposed that L is imbedded in a metric space and (0, 1) denotes the interval $0 \leqq y \leqq 1$ of the real numbers. A continuum is *cyclic* if it has no cut point.

[6] See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, chap. 5; note also references to P. Alexandroff and to C. Kuratowski.

[7] See Whyburn, *Non-alternating interior retracting transformations*, Annals of Mathematics, (2), vol. 40 (1939), pp. 914–921.

is open in $a'b'$ and $h^{-1}g(V)\supset x$ and is open in M. Since f is interior at x, $fh^{-1}g(V)$ has $f(x)$ as an interior point; and since this set surely is contained in $fh^{-1}gh(U)=\phi(U)$, it follows that $\phi(x)$ is interior to $\phi(U)$. If $x\varepsilon L-M$, we may suppose $U\subset L-M$. Then h is topological on U, so that $h(U)$ is open in L'. Thus $gh(U)$ is open in $a'b'$, since g is interior on L'. Hence $h^{-1}gh(U)$ is open in M and $fh^{-1}gh(U)=\phi(U)$ has $\phi(x)$ as an interior point, since f is interior at x.

It is clear that essentially the same argument suffices to establish the following somewhat more general extension theorem:

Let M be a subcontinuum of a locally connected continuum L of the form $L=C(a, b)$ where $a, b\varepsilon M$. Let $f(M)=(0, 1)$ be such that $f(a)=0$, $f(b)=1$ and for any y with $0<y<1$, every component of $L-f^{-1}(y)$ contains either a or b. There exists an extension $\phi(L)=(0, 1)$ of f to L which is interior at every point of $L-M$ and also at every point of M where f is interior.

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