ON MAJORANTS OF SUBHARMONIC AND ANALYTIC FUNCTIONS

FRANTIŠEK WOLF

This paper represents a different approach to a whole group of problems connected with majorants of subharmonic functions. The same method has been used previously in order to prove a generalization of the Phragmén-Lindelöf theorem.[1] It seems that the best approach is to prove first Lemma 4, and then the most important results are easily deducible. Corollary 6 is a generalization of a result of N. Levinson.[2] His theorem has made me realize the importance of these results.

LEMMA 1. *If* (i) $0<f(x)\leqq 1$ *and* (ii) $\int_a^b \log f(x)\cdot dx$ *is finite, then*

$$\int_a^b \log \left| \int_\xi^x f(y)dy \right| dx \tag{1}$$

is a continuous function of ξ *in* (a, b).

We first suppose that $f(x)$ is non-decreasing and that $(0, 1)=(a, b)$. We get

$$\int_0^x f(y)dy > \int_{x/2}^x f(y)dy \geqq (x/2)f(x/2).$$

Hence

$$\begin{aligned}\int_0^1 \log\left(\int_0^x f(y)dy\right)dx &> \int_0^1 \log\,(x/2)dx + \int_0^1 \log f(x/2)dx \\ &> 2\int_0^1 \log x\cdot dx + 2\int_0^1 \log f(x)dx \\ &= -2 + 2\int_0^1 \log f(x)\cdot dx.\end{aligned} \tag{2}$$

If $f(x)$ is replaced by $f(a+(b-a)x)$, we obtain

Presented to the Society, November 22, 1941 under the title *On majorants of analytic functions*; received by the editors March 2, 1942.

[1] Cf. the end of this paper and Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[2] *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, p. 127, Theorem 43.

$$\int_a^b \log\left(\int_a^x f(y)dy\right)dx \geqq -2(b-a) + (b-a)\log(b-a) + 2\int_a^b \log f(x)\cdot dx. \tag{3}$$

Next, if $f(x)$ is general, we form the rearranged non-decreasing function $\bar{f}(x)$ for which

$$\operatorname{meas} \underset{x}{E}\,[f(x) \leqq y] = \operatorname{meas} \underset{x}{E}\,[\bar{f}(x) \leqq y]$$

for all y. We know that[3]

$$\int_0^1 \log f(x)\cdot dx = \int_0^1 \log \bar{f}(x)\cdot dx,$$

and that

$$\left|\int_{x_0}^x f(y)\cdot dy\right| \geqq \int_0^{|x-x_0|} \bar{f}(y)\cdot dy.$$

Hence, if $x_0 \subset (0, 1)$,

$$\begin{aligned}
&\int_0^1 \log\left|\int_{x_0}^x f(y)\cdot dy\right|\cdot dx \\
&\geqq \int_0^1 \log\left(\int_0^{|x-x_0|} \bar{f}(y)\cdot dy\right)\cdot dx \\
&= \int_0^{x_0} \log\left(\int_0^{x_0-x} \bar{f}(y)\cdot dy\right)\cdot dx + \int_{x_0}^1 \log\left(\int_0^{x-x_0} \bar{f}(y)\cdot dy\right)dx \\
&= \int_0^{x_0} \log\left(\int_0^x \bar{f}(y)\cdot dy\right)\cdot dx + \int_0^{1-x_0} \log\left(\int_0^x \bar{f}(y)dy\right)dx \\
&\geqq 2\int_0^1 \log\left(\int_0^x \bar{f}(y)\cdot dy\right)\cdot dx.
\end{aligned}$$

This, with (2) gives

$$\begin{aligned}
\int_0^1 \log\left(\int_{x_0}^x f(y)\cdot dy\right)\cdot dx &\geqq -4 + 4\int_0^1 \log \bar{f}(x)\cdot dx \\
&= -4 + 4\int_0^1 \log f(x)\cdot dx.
\end{aligned}$$

---

[3] Hardy, Littlewood, Pólya, *Inequalities*, Cambridge, 1934, p. 276.

In a similar fashion we get, for $x_0 \subset (a, b)$,

$$\int_a^b \log \left| \int_{x_0}^x f(y)dy \right| dx \geqq -4(b-a) + 2(b-a)\log(b-a) + 4\int_a^b \log f(x)\cdot dx. \tag{4}$$

If $\epsilon > 0$ is given, we can find a $\delta > 0$ such that

$$8\delta - 4\delta \log \delta - 4\int_{x_0-\delta}^{x_0+\delta} \log f(x)\cdot dx < \epsilon/3.$$

Let us denote by J_1 the common part of (a, b) and $(x_0-\delta, x_0+\delta)$, and by J_2 the rest of (a, b). Then, by (3), for $\xi \subset J_1$

$$0 > \int_{J_1} \log \left| \int_\xi^x f(y)\cdot dy \right| \cdot dx \geqq -8\delta + 4\delta \log \delta + 4\int_{x_0-\delta}^{x_0+\delta} \log f(x)\cdot dx > -\epsilon/3.$$

Further

$$g_2(\xi) = \int_{J_2} \log \left| \int_\xi^x f(y)\cdot dy \right| \cdot dx$$

is a continuous function for $\xi = x_0$ and, hence, there is a $\delta_1 < \delta$ such that

$$| g_2(\xi) - g_2(x_0) | \leqq \epsilon/3 \qquad \text{for } | \xi - x_0 | < \delta_1.$$

Hence, if we call the integral (1) $g(\xi)$, then

$$| g(\xi) - g(x_0) | < \epsilon \qquad \text{for } | \xi - x_0 | < \delta_1.$$

This shows that (1) is a continuous function of ξ at an arbitrary point $x_0 \subset (a, b)$.

LEMMA 2. *Given a non-negative $\psi(x) \subset L$, there is a domain D, bounded by two continuous curves*

$$C_1 \equiv y = g_1(x), \qquad C_2 \equiv y = g_2(x)$$

and two straight lines $x = x_0 - a$, $x = x_0 + a$, such that, if $x + iy = f(re^{i\theta})$ represents D, conformally on the unit circle $r < 1$, then

$$\left| \frac{dx}{d\theta} \right|_{r=1} \geqq \text{const. exp } [\psi(x)] \tag{5}$$

on C_1 and C_2.

Further the rectangle

(6) $$R_0: [|y - y_0| < a \cdot \log 3/2, \ |x - x_0| < a]$$

is interior to D and

$$|y - y_0| \leqq c(a), \qquad |x - x_0| \leqq a, \qquad \textit{with} \lim_{a \to 0} c(a) = 0,$$

contains D.

There is no loss of generality in supposing $x_0 = y_0 = 0$. We define D by constructing its conformal representation on the unit circle. First we define the boundary function of the harmonic function $x(r, \theta)$ so that it satisfies the above condition for the derivative. We define it as the inverse function of

(7) $$\theta(x) = b \int_{-a}^{x} e^{-\psi(t)} dt$$

and, for later convenience, we take b such that

(8) $$b \int_{-a}^{a} e^{-\psi(t)} dt = \theta_0 < 2 \text{ arc cos } 3/4 < \pi/3.$$

Then the inverse function $x(1, \theta)$, defined in $(0, \theta_0)$ satisfies condition (5) and is continuous and decreasing. Further, we define $x(1, \theta) = a$, for $\theta \subset (\theta_0, \pi)$; $x(1, \pi+\theta) = x(1, \theta_0 - \theta)$ for $\theta \subset (0, \theta_0)$, and finally $x(1, \theta) = -a$ for $\theta \subset (\pi+\theta_0, 2\pi)$. In the interval $(\pi, \pi+\theta_0)$ condition (5) is again satisfied. Now, the boundary function is completely defined and

$$x(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{x(1, \phi) d\phi}{1 - 2r \cos(\theta - \phi) + r^2}.$$

The conjugate harmonic function is

$$y(r, \theta) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} x(1, \phi) d\phi.$$

By partial integration

$$y(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - re^{i(\theta - \phi)} \right| dx(1, \phi).$$

Since x is constant in some intervals, we get, using the above definition of $x(1, \phi)$,

$$(9)\quad y(r,\theta)=\frac{1}{2\pi}\int_0^{\theta_0}\left[\log\left|1-re^{i(\theta-\phi)}\right|-\log\left|1+re^{i(\theta-\theta_0+\phi)}\right|\right]dx(1,\phi).$$

Now, the required conformal representation is given by $x(r, \theta)+iy(r, \theta)$. The boundary curve of D in the parametric form is $x(1, \theta)+iy(1, \theta)$, and we shall investigate it.

If $\theta\subset(0, \theta_0)$, then $x(1, \theta)$ runs from $-a$ to $+a$ and, since $\left|1-e^{i(\theta-\phi)}\right|<1$ for $\theta, \phi\in(0, \theta_0)\subset(0, \pi/3)$ (cf. (8)),

$$\begin{aligned} y(1,\theta) &\leqq -\frac{1}{2\pi}\int_0^{\theta_0}\log\left|1+e^{i(\theta-\theta_0+\phi)}\right|dx \\ &\leqq -\frac{1}{2\pi}\log\left|1+e^{i\theta_0}\right|\int_{-a}^{a}dx \\ &= -(a/\pi)\log 2\cos\theta_0/2 < -(a/\pi)\log 3/2. \end{aligned}$$

The first term I_1 on the right in (9) is more difficult to dispose of. We have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_0^{\theta_0}\log\left|2\sin\frac{\theta-\phi}{2}\right|\cdot dx(1,\phi) \\ &= \frac{1}{2\pi}\int_0^{\theta_0}\log\frac{2\sin(\theta-\phi)/2}{\theta-\phi}\cdot dx(1,\phi) \\ &\quad+\frac{1}{2\pi}\int_0^{\theta_0}\log\left|\theta-\phi\right|\cdot dx(1,\phi) = I_1' + I_1'', \end{aligned}$$

Here, I_1' is evidently a continuous function of θ and

$$0 > I_1 > -\frac{a}{\pi}\log\frac{2\sin\theta_0/2}{\theta_0}\cdot$$

By means of (7), we change variables in I_1'' and obtain

$$I_1''=\frac{1}{2\pi}\int_{-a}^{a}\log\left|b\int_{\xi}^{x(\theta)}e^{-\psi(\alpha)}\cdot d\alpha\right|\cdot d\xi.$$

By Lemma 2, this is a continuous function of x and therefore also of θ and from (3) we get

$$0 > I_1'' > -\frac{2a}{\pi}+\frac{a\log 2a}{\pi}+\frac{1}{\pi}\int_{-a}^{a}\log\left|be^{-\psi(x)}\right|dx.$$

If we combine the last inequalities and introduce the abbreviation $c(a)$, we get

$$-\frac{a}{\pi}\log 3/2 > y(1, \theta) > -c(a), \qquad \theta \subset (0, \theta_0),$$

with

$$\lim_{a\to 0} c(a) = 0.$$

From (9) it is easy to deduce that

$$y(r, \theta+\pi) = -y(r, \theta_0-\theta),$$

and we obtain

$$(a/\pi)\log 3/2 < y(1, \theta) \leqq c(a), \quad \text{for } \theta \subset (\pi, \pi+\theta_0).$$

Using the same notation we shall prove the following lemma.

LEMMA 3. *If $\psi(x_0-a)$, $\psi(x_0+a)$ are finite, then every subharmonic function $\alpha(x, y)$, defined in D, which satisfies*

$$\alpha(x, y) \leqq e^{\psi(x)}, \qquad |x| < a, \tag{10}$$

has an upper bound in $R[|y| < a\cdot\log 3/2,\ |x| < a]$ which depends only on $\psi(x)$.

If we represent D conformally on the unit circle C, we get from $\alpha(x, y)$ a subharmonic function $\alpha(r, \theta)$ defined in C. There, it must be less than any harmonic function with boundary values not less than those of $\alpha(r, \theta)$. By (10), these are not greater than $\exp\{\psi(x(1, \theta))\}$, where $x(1, \theta)$ has been defined in the preceding lemma. Such a harmonic function is the Poisson integral of $\exp\{\psi(x(1, \theta))\}$. We have to show that this Poisson integral is not identically equal to ∞. A sufficient condition is

$$\int_0^{2\pi} \exp\{\psi(x(1, \theta))\}\cdot d\theta < \infty.$$

In view of the definition of $x(1, \theta)$, this integral is obviously less than

$$\left|\int_{-a}^{a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\theta_0}^{\pi} e^{\psi(a)}d\theta + \left|\int_{a}^{-a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\pi+\theta_0}^{2\pi} e^{\psi(-a)}d\theta,$$

and using (5), also less than

$$2b\int_{-a}^{a} e^{\psi(x)}e^{-\psi(x)}dx + \pi(e^{\psi(a)} + e^{\psi(-a)}) \leqq 4ab + \pi(e^{\psi(a)} + e^{\psi(-a)}) < \infty.$$

Hence $\alpha(x, y)$ is, inside of D, less than a finite harmonic function depending only on $\psi(x)$. In the domain D, completely interior to R, it is therefore bounded from above by a constant depending only on $\psi(x)$.

THEOREM. *If* (i) *$\alpha(x, y)$ is subharmonic in $R[|x|<c, |y|<d]$ and*

$$\text{(ii)} \quad \alpha(x, y)<e^{\psi(x)}, \ \psi(x)\subset L, \ x\subset(-c, c), \ \psi(-c)<\infty, \ \psi(c)<\infty,$$

then for any δ, such that $0<\delta<d$, there is an upper bound C for $\alpha(x, y)$ in $D_0[|x|<c, |y|<d-\delta]$ dependent only on δ and $\psi(x)$, but independent of the particular $\alpha(x, y)$.

In view of the Lemma 3, it is sufficient to show that there is an a and a finite number of domains D_k satisfying the conditions of this lemma, and, such that D_k contains (cf. (6)) $R_k[|x-x_k|<a, |y-y_k| \leqq c(a)]$, is contained in R, and that $\sum R_k$ covers completely D_0. Since $\alpha(x, y)$ has a finite upper bound in every R_k, it must be so also in D_0. And the upper bound will depend only on $\psi(x)$ and D_0.

We determine $a>0$ such that $c(a)-a \log 3/2<\delta/2$. If we define $D^*[|x|<c, |y|<d-\delta/2]$, then $D_0\subset D^*\subset R$, and, if $R_k\subset D^*$, then the corresponding $D_k\subset R$ (cf. (3.2)). Since R_k can be any rectangle, of the above size, in D^* and such that $\psi(x_k\pm a)$ is finite, it is evident that we can find a finite number of them covering completely D_0. The theorem is proved.

COROLLARY: *Let $f(z)$ be a function, analytic in $R[|x|<c, |y|<d]$, such that*

$$|f(z)| \leqq M(x).$$

If $\int_{-c}^{c}\log^+\log^+ M(x)dx<\infty$, then for every domain D_0 completely interior to R, there exists a ϕ depending only on D_0 and $M(x)$, such that

$$|f(z)| \leqq \phi \qquad \textit{for } z \subset D_0.$$

There is no loss of generality to suppose $M(x)>e$. and then the sign $+$ can be omitted over the log signs. If $f(z)$ is analytic, then $\log|f(z)|$ is subharmonic and the result follows from the preceding theorem.

Nils Sjöberg has proved the following theorem:[4]

Let $M(\theta)$ be given and $0<\epsilon<1$. In order that the class of subharmonic functions, defined in $|z|<1$, which satisfy in $1-\epsilon\leqq|z|<1$

$$|\mu(re^{i\theta})| \leqq M(\theta),$$

[4] Comptes Rendus du Congrès des Mathématiques à Helsinfors, 1938.

should be bounded from above in every circle $|z| \leqq r_0 < 1$, *it is sufficient that*

$$\int_{-\pi}^{\pi} \log^+ M(\theta) \cdot d\theta < \infty.$$

This theorem can easily be deduced from our result. By $\zeta = \log z$, we straighten out the concentric circles, and $1-\epsilon \leqq |z| < 1$ corresponds to $\log (1-\epsilon) \leqq R(\zeta) < 0$. In this strip $\mu(\zeta) \leqq M(\theta)$, $\theta = \Im(\zeta)$. Hence, if $M(\theta_0) < \infty$, then, by our result, the class of subharmonic functions $\mu(\zeta)$ will be bounded in $\theta_0 \leqq \arg z = \theta \leqq \theta_0 + 2\pi$ and $\log (1-\epsilon/2) \leqq R(\zeta) \leqq \log (1-\epsilon/4)$. Hence the same is true of $\mu(z)$ in $1-\epsilon/2 \leqq |z| \leqq 1-\epsilon/4$. But the class of subharmonic functions must have the same upper bound in $|z| \leqq 1-\epsilon/4$. The theorem is proved.

Similarly we can prove a generalization of the Phragmén-Lindelöf theorem:[5]

If: (i) $f(z)$ *is analytic in* $\Im(z) > 0$, (ii) *its boundary values on* $\Im(z) = 0$ *are in absolute value less than* 1, (iii) *there are two sequences* $r_k \to \infty$ *and* $\epsilon_k \to 0$ *such that*

$$|f(z)| \leqq \exp\{\epsilon_k r_k e^{\psi(\theta)}\}, \qquad \psi(\theta) \subset L,$$

for $r_k(1-\delta) < |z| < r_k$, *then*

$$|f(z)| \leqq 1 \qquad \text{for } \Im(z) > 0.$$

From condition (iii) it follows that

$$\log |f(r_k z)| / \epsilon_k r_k \leqq e^{\psi(\theta)},$$

for $1-\delta < |z| < 1$. By condition (ii), we can suppose $\psi(0) = \psi(\pi) = 0$. In the same way as in the preceding we deduce the existence of a ϕ, such that

$$\log |f(r_k z)| / \epsilon_k r_k < \phi \qquad \text{for } 1 - \delta/2 < |z| < 1 - \delta/4,$$

or

$$|f(z)| \leqq \exp\{\epsilon_k \phi r_k\}, \qquad (1 - \delta/2) r_k < |z| < (1 - \delta/4) r_k.$$

Now we use the Phragmén-Lindelöf theorem in its classical form[6] to deduce the desired result.

University of California

[5] Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[6] R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936, p. 43.