THE CAUCHY THEOREM FOR FUNCTIONS ON CLOSED SETS

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The object of this paper is to extend the theorem of Cauchy to functions of a complex variable defined on any bounded closed set, E, by determining conditions on $f(z)$ in order that for certain coverings of E, C_n, and an extension of $f(z)$, $f^*(z)$, $\lim_{n\to\infty} \int_{c_n} f^*(z)dz=0$. It was suggested partly by the notion of a general monogenic function due to Trjitzinsky[1] and partly by the measure theory methods of Menchoff[2] and others, which succeed so well in lightening the restrictions on the real and imaginary parts of a complex function in order that $f(z)$ be regular.

Throughout this paper we shall consider only rectangles with sides parallel to the real and imaginary axes. A C-covering of a plane set F, denoted by C, will be a set of closed rectangles, possibly abutting, but nonoverlapping, which contain F. c will denote the boundary of C. The covering C_n is to be composed of rectangles R_{mn} so that $C_n=\sum_m R_{mn}$ $(m, n=1, 2, \cdots)$.

1. **The extension,** $f^*(z)$. If $u(P)$ is a positive continuous function defined on the closed and bounded set F in the plane, we shall let[3] $u^*(P)=\max_{Q\in F} u(Q)\{2-d(P, Q)/d(P, F)\}$ for P not in F, and $u^*(P)=u(P)$ for P in F, where $d(P, Q)$ denotes the distance from P to Q and $d(P, F)$ the distance from the set F to P. In general, if $u(P)$ is continuous, since $u(P)=(u(P)+|u(P)|)/2-(|u(P)|-u(P))/2$, that is, since $u(P)$ is the difference of two continuous positive functions, $u^*(P)$ will denote the extension of $u(P)$ obtained by extending as before these parts. If $f(z)$ $(=u(x, y)+iv(x, y))$ is defined on a bounded closed set and continuous, $f^*(z)$ will denote $u^*(x,y)+iv^*(x,y)$.

LEMMA 1. *If $u(P)$ is defined on a bounded closed set F and $|u(Q)-u(P)|<M(P)d(P, Q)$ where $M(P)$ is a finite function of P defined on F, then $|u^*(P)-u^*(Q)|<20\ M(P)\ d(P, Q)$, for P in F and Q arbitrary.*

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[1] W. J. Trjitzinsky, *Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux*, Annales Scientifique de L'École Normale Supérieure, Paris, 1938, p. 120.

[2] D. Menchoff, *Les Conditions de Monogénéité*, Actualités Scientifiques et Industrielles, no. 329, Paris, 1936.

[3] S. Bochner, *Fourier Lectures*, 1936–1937, Princeton, p. 62.

PROOF. Consider first the case for which $u(P) \geqq 0$. If Q is any point not in D, if Q_0 is a point in F for which $d(Q, Q_0) = d(Q, F)$, and if, of the points S satisfying the inequality $d(S, Q) \leqq 2d(Q, Q_0)$, R is the point where the maximum of u is attained, then $u^*(Q_0) \leqq u^*(Q) \leqq u^*(R)$. Hence

$$\begin{aligned} |u^*(Q) - u^*(P)| &\leqq |u^*(Q) - u^*(R)| + |u^*(R) - u^*(P)| \\ &\leqq |u^*(Q_0) - u^*(P)| + 2|u^*(R) - u^*(P)| \\ &< M(P)d(Q_0, P) + 2M(P)d(R, P). \end{aligned}$$

It is easily verified that $d(Q_0, P) \leqq 2d(Q, P)$ and $d(R, P) \leqq 4d(Q, P)$, so that $|u^*(Q) - u^*(P)| < 10M(P)\, d(P, Q)$ and the lemma is proved for case of $u(P)$ positive. In the general case, $u(P) = (|u| + u)/2 - (|u| - u)/2 = g(P) - h(P)$, where g and h are positive functions, and satisfy the conditions of the lemma, so that for P in F and Q arbitrary $|g^*(Q) - g^*(P)|$ and $|h^*(Q) - h^*(P)|$ are each less than $10M(P)\, d(P, Q)$. Hence for $u^* = g^* - h^*$, it readily follows that $|u^*(Q) - u^*(P)| < 20M(P)\, d(P, Q)$, and the proof of the lemma is complete.

2. **Bounded derivatives.** We shall use the following fundamental lemma:[4]

LEMMA 2. *Let $w(x, y)$ be a real, continuous function defined in the square S, the sides of which are parallel to the coordinate axes, and let F be a closed set in S and such that*

$$\begin{aligned} |w(x + h, y) - w(x, y)| &\leqq M|h|, \\ |w(x, y + k) - w(x, y)| &\leqq M|k| \end{aligned}$$

for all points (x, y) in F and for all points $(x+h, y)$, $(x, y+k)$ of the square S, where M is a constant. Finally let R be the least rectangle with sides parallel to the axes containing F.[5]

Under these conditions[6] *the following inequalities hold:*

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)]dx - \iint_F \frac{\partial w}{\partial y}\, dxdy \right| \leqq 5Mm(S - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)]dy - \iint_F \frac{\partial w}{\partial x}\, dxdy \right| \leqq 5Mm(S - F)$$

where (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_1, y_2), $(x_1 \leqq x_2,\ y_1 \leqq y_2)$ are the

[4] For the proof of this lemma, cf. loc. cit., p. 10.

[5] The "least rectangle" may be only a segment or a point.

[6] The conditions on w imply the existence of the partial derivatives a.e. in F.

corners of the rectangle R and $m(S-F)$ is the measure of the set $S-F$, which is composed of points of S not in F.

THEOREM 1. *Let $f(z)$ $(=u(x, y)+iv(x, y))$ be defined on the bounded closed set E, and let R be a rectangle with sides parallel to the axes containing at least one point of E on each side. If (letting $F=E\cdot R$)*

(1) *for all z and $z+h$ in F, and a constant B,*

$$\left|\frac{f(z+h)-f(z)}{h}\right| < B,$$

(2) *the Cauchy-Riemann equations hold a.e. (almost everywhere) in F, where the partial derivatives of u and v exist,[7] then $\left|\int_r f^*(z)dz\right| < 400Bm(S-F)$ where r is the boundary of R, and S is a square of least area containing R.*

PROOF. If $h=k+il$, condition (1) implies

$$\left|\frac{u(x+k, y+l)-u(x, y)}{h}\right| < B,$$

and a similar condition on $v(x, y)$, for every point z, and $z+h$, in F. According to Lemma 1,

$$\left|\frac{u^*(x+k, y+l)-u^*(x, y)}{h}\right| < 20B,$$

for each point z in F, and $z+h$ in R. Hence by Lemma 2,

$$\left|\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx - \iint_F \frac{\partial u^*}{\partial y}\,dxdy\right| < 100Bm(S-F),$$

(x_1, y_1) and (x_2, y_2) being corners of R. Similar inequalities for $u^*(x, y)$ with respect to y, and $v^*(x, y)$ with respect to x and y also hold. But

$$\int f^*(z)dz = \int_r u^*dx - v^*dy + i\int_r v^*dx + u^*dy$$

and

$$\int_r u^*dx = -\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx.$$

[7] This is in no way a further restriction on E, for almost all points of any measurable plane set are points of linear density for it both in the direction of the x-axis and in that of the y-axis.

The condition that the limits, $\lim_{h\to 0}(f(z+h)-f(z))/h$, as $z+h$ approaches z through points of E along either of two curves having non-collinear tangents at z, should be equal, is equivalent to the condition (2) in the presence of (1). For (1) and (2) imply that $f^*(z)$ is monogenic a.e. in E (Menchoff, loc. cit., Theorem 2, p. 27 and Theorem 5, p. 23).

Let $\epsilon' = \int_{x_1}^{x_2} [u^*(x, y_2) - u^*(x, y_1)]\, dx - \iint_F (\partial u^*/\partial y) dy dx$. Then $\int_r u^* dx = -\iint_F (\partial u^*/\partial y) dy dx - \epsilon'$. Taking into account similar reasoning for the other parts of $\int_r f^*(z) dz$ we have

$$\int_r f^*(z) dz = -\iint_F \left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right) dx dy + i \iint_F \left(\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y} \right) dx dy + \epsilon$$

where $|\epsilon| < 400 B m(S-F)$. Since the Cauchy-Riemann equations hold a.e. in F, $\int_r f^*(z) dz = \epsilon$, and the proof of the theorem is complete.

COROLLARY. *Let $f(z)$ be defined on the bounded closed set E with a bounded derivative there. Let $S = \sum_m S_m$ be a C-covering of E by squares with R_m the least rectangle within S_m containing $S_m \cdot E$. Then there is a constant B for which $\sum_m |\int_{r_m} f^*(z) dz| < 400 B m(S-E)$, and if $S \rightarrow E$, $C = \sum_m R_m \rightarrow E$, and* $\lim_{C \rightarrow E} \int_c f^*(z) dz = 0$.

3. Derivatives finite, except for a denumerable set. We prove this theorem:

THEOREM 2. *If $f(z)$ is defined and continuous on the bounded closed set E, and if, except for a denumerable number of points,* $\limsup_{h \rightarrow 0} |(f(z+h) - f(z))/h| < \infty$, *and the Cauchy-Riemann equations hold a.e. where the partial derivatives of u and v exist, then there is a sequence of C-coverings, $\{C_n\}$, for which* $\lim_{n \rightarrow \infty} \sum_m |\int_{r_{mn}} f^*(z) dz| = 0$.

PROOF. Define $I(C) = \sum_m |\int_{r_m} f^*(z) dz|$. If for every point z of E there is a neighborhood $N(z)$ such that for every closed subset of E in N, there is a sequence of coverings $\{C_n\}$ for which $\lim_{n \rightarrow \infty} I(C_n) = 0$, then by the Heine-Borel theorem there exists a sequence of coverings of E with the property mentioned in the theorem. The proof will be complete therefore, if we show that there is such a neighborhood for each point of E. Let P be those points of E such that in every neighborhood of z there is a closed subset of E for which there is no sequence of C-coverings, $\{C_n\}$, for which $\lim_{n \rightarrow \infty} I(C_n) = 0$. We shall assume that P is not empty and show that this leads to an absurdity.

Let P_m $(m = 1, 2, \cdots)$ be the points of P for which each of the absolute values,

$$|u^*(x+k, y) - u^*(x, y)|, \quad |v^*(x+k, y) - v^*(x, y)|,$$
$$|u^*(x, y+k) - u^*(x, y)|, \quad |v^*(x, y+k) - v^*(x, y)|$$

is less than or equal to $m|k|$ for $|k| \leqq 1/m$, k a real number. Since

u^* and v^* are continuous and P is closed, P_m is closed. Since at each point of E, except for a denumerable set H, the partial derivates are finite, $P=\sum_m P_m+P\cdot H$. By Baire's[8] theorem, there is an isolated point of P in H, or for some integer N there is a point z_0 in P, the center of a square S which contains only points of P which are in P_N. The former alternative is quickly dismissed as impossible; we proceed on the basis of the latter, and let F be any closed subset of $E\cdot S$. Subdivide the sides of S into n equal parts, $n>2N$, and obtain the squares S_j $(j=1, 2, \cdots, n^2)$. ϵ being given, choose n so great that the squares $\overline{S}_j$ which contain points of $F\cdot P$ satisfy the inequality, $m(\sum_j \overline{S}_j-P\cdot S)<\epsilon/800N$. If $\overline{R}_j$ is the least rectangle containing $P\cdot\overline{S}_j$, and $\overline{C}$ is the covering $\sum_j \overline{R}_j$, by Theorem 1, $I(\overline{C})<400\, N\sum_j m(\overline{S}_j-P\cdot\overline{S}_j)<\epsilon/2$. Since $I(R)$ is a continuous function of r, $\overline{C}$ may be extended by the addition of more small rectangles, so that, if C' is the new covering, $I(C')$ remains less than $\epsilon/2$, but so that the points of $F\cdot P$ are inner points of the covering. The part of F not already covered (denote it by G) is such that its closure contains only points z of F for which there is some neighborhood $N(z)$ with the property that every closed subset of F in N can be C-covered, say by $C_n(z)$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty} I(C_n(z))=0$. Let $S(z)$ be a square with z as center entirely within $N(z)$. Of these squares a finite number, k, cover G, and within each of these is a covering, $C(z)$, of G for which $I(C(z))<\epsilon/2k$. Hence G is C-covered by a covering C for which $I(C)<\epsilon/2$. F is therefore C-covered by $C+C'$ for which $I(C+C')<\epsilon$, so that z_0 cannot belong to P, contrary to assumption. This completes the proof of Theorem 2.

COROLLARY. *If $f(z)$, defined on the bounded closed set E and continuous there, has a derivative at each point except at most a denumerable set, there is a sequence of C-coverings of E with E as their limit for which* $\lim_{n\to\infty}\int_{C_n} f^*(z)dz=0$.

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[8] S. Saks, *The Theory of the Integral*, New York, 1937, p. 54.