DUAL GEODESICS ON A SURFACE

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Introduction. Union curves and dual union curves have been defined and studied in projective space by Sperry.[1] It is well known that the union curves of the congruence of normals to a metric surface are the geodesics on the surface. The principal aim of this note is to obtain the differential equation of the *dual* geodesics on a metric analytic surface in ordinary space.

The notation of Eisenhart[2] will be employed for the most part. However, $\Gamma^{\alpha}_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will take the range 1, 2, and Latin indices the range 1, 2, 3.

1. **Ray-point corresponding to a point of a curve on the surface.** The tangent planes to the surface S $(x^i=x^i(u^1, u^2))$ at the point $P(x^i)$ and at two "successive" points of the curve C $(u^\alpha=u^\alpha(s))$ on S are given by

$$
\begin{aligned}
&(\xi^i - x^i)X^i = 0,\\
&(\xi^i - x^i)\frac{\partial X^i}{\partial u^\alpha}u'^\alpha = 0,\\
&(\xi^i - x^i)\left(\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta}u'^\alpha u'^\beta + \frac{\partial X^i}{\partial u^\alpha}u''^\alpha\right) = \frac{\partial X^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}u'^\alpha u'^\beta,
\end{aligned}
\tag{1}
$$

where the primes indicate differentiation with respect to s.

The ray-point[3] R of the curve C corresponding to the point P is the point of intersection of the three planes (1). The coordinates of R are given by

$$
S(\xi^i - x^i) = \delta^{jk}_{\sigma\nu}X^\sigma \frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta}\frac{\partial X^\nu}{\partial u^\gamma}u'^\alpha u'^\beta u'^\gamma, \tag{2}
$$

where i, j, k take the values 1, 2, 3 cyclically, $\delta^{jk}_{\sigma\nu}$ is a Kronecker delta, and S is defined by

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[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1928), p. 213.

[2] Eisenhart, *Differential Geometry*, Princeton University Press, 1940.

[3] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932.

$$S \equiv \delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} u'^\gamma \left(\frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta + \frac{\partial X^k}{\partial u^\alpha} u''^\alpha \right). \tag{3}$$

The tangents to the parametric curves at P, and the normal to the surface at P constitute a local system of reference. In order to deduce the local coordinates of the point R from equations (2), we employ the formulas from Eisenhart,[4]

$$\frac{\partial X^k}{\partial u^\alpha} = - d_{\alpha\gamma} g^{\gamma\sigma} \frac{\partial x^k}{\partial u^\sigma}, \tag{4}$$

$$\frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} = \frac{\partial X^k}{\partial u^\gamma} \overline{\Gamma}^\gamma_{\alpha\beta} - h_{\alpha\beta} X^k, \tag{5}$$

$$\overline{\Gamma}^\mu_{\alpha\beta} = d^{\nu\mu} \frac{\partial d_{\alpha\nu}}{\partial u^\beta} - d^{\nu\mu} d_{\alpha\gamma} \Gamma^\gamma_{\nu\beta}; \tag{6}$$

and we use the fact that

$$\delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^1} \frac{\partial x^k}{\partial u^2} = g^{1/2} \tag{7}$$

to obtain for the first part of the expression for S

$$\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} \frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta u'^\gamma = g^{1/2} (g^{\tau 1} g^{\epsilon 2} - g^{\tau 2} g^{\epsilon 1}) d_{\gamma\tau} d_{\mu\epsilon} \overline{\Gamma}^\mu_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma. \tag{8}$$

On summing out τ and ϵ, and on writing $|g^{\alpha\beta}| = g^{-1}$, the right member of (8) reduces to

$$g^{-1/2} \delta^{\sigma\tau}_{12} d_{\gamma\sigma} d_{\mu\tau} \overline{\Gamma}^\mu_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma. \tag{9}$$

In consequence of (4), we have for the second part of the expression for S

$$\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\alpha} \frac{\partial X^k}{\partial u^\beta} u'^\alpha u''^\beta = \delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^\sigma} \frac{\partial x^k}{\partial u^\tau} d_{\alpha\gamma} d_{\beta\delta} g^{\gamma\sigma} g^{\delta\tau} u'^\alpha u''^\beta. \tag{10}$$

On summing σ and τ, and on making use of (7), the right member of (10) takes the form

$$g^{1/2} (g^{\gamma 1} g^{\delta 2} - g^{\gamma 2} g^{\delta 1}) d_{\alpha\gamma} d_{\beta\delta} u'^\alpha u''^\beta, \tag{11}$$

which, in turn, reduces to

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[4] Eisenhart, loc. cit., p. 217, p. 257.

$$g^{-1/2}\delta_{12}^{\sigma\tau}d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}. \tag{12}$$

Because of (9) and (12), equation (3) takes the form

$$S \equiv g^{-1/2}\delta_{12}^{\sigma\tau}(d_{\gamma\sigma}d_{\mu\tau}\overline{\Gamma}^{\mu}_{\alpha\beta}u'^{\alpha}u'^{\beta}u'^{\gamma} + d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}). \tag{13}$$

In order to express the right member of (2) as a linear combination of the partial derivatives $\partial x^i/\partial u^\alpha$, we use the definition

$$\frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta} = -d_{\alpha\beta}, \tag{14}$$

and also the formulas from Eisenhart[5]

$$\begin{aligned}\delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^1} &= eh^{1/2}h^{2\alpha}\frac{\partial X^i}{\partial u^\alpha},\\ \delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^2} &= -eh^{1/2}h^{1\alpha}\frac{\partial X^i}{\partial u^\alpha},\end{aligned} \tag{15}$$

where i, j, k take the values 1, 2, 3 cyclically, and where e is $+1$ or -1 according as the Gaussian curvature is positive or negative. In consequence of (14), (15), and (4), the right member of (2) assumes the form

$$eh^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}(u'^{1}h^{2\sigma} - u'^{2}h^{1\sigma})d_{\sigma\tau}g^{\tau\delta}\frac{\partial x^i}{\partial u^\delta}. \tag{16}$$

If we multiply the equation[6]

$$h_{\alpha\beta} = d_{\alpha\gamma}d_{\beta\delta}g^{\gamma\delta} \tag{17}$$

by $d^{\alpha\tau}$ and sum on α, and then multiply the resulting equation by $h^{\beta\sigma}$ and sum on β, we obtain

$$d^{\sigma\tau} = h^{\beta\sigma}d_{\beta\delta}g^{\tau\delta}. \tag{18}$$

Because of (18), the expression (16) may be written in the form

$$eh^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}\delta_{\gamma\sigma}^{12}u'^{\gamma}d^{\sigma\delta}\frac{\partial x^i}{\partial u^\delta}, \tag{19}$$

which is the expression desired for the right member of equation (2). Use of (13) and (19) in (2) yields the coordinates ξ^i of the ray-point R of the curve C $(u^\alpha = u^\alpha(s))$ at the point $P(x^i)$ in the form

[5] Eisenhart, loc. cit., p. 259.
[6] Eisenhart, loc. cit., p. 253.

$$(20)\quad \xi^i = x^i + e(hg)^{1/2} d_{\alpha\beta} u'^{\alpha} u'^{\beta} \frac{\delta^{12}_{\gamma\sigma} u'^{\gamma} d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma} d_{\mu\tau} \overline{\Gamma}^{\mu}_{\alpha\beta} u'^{\alpha} u'^{\beta} u'^{\gamma} + d_{\alpha\sigma} d_{\beta\tau} u'^{\alpha} u''^{\beta})} \cdot \frac{\partial x^i}{\partial u^{\delta}}.$$

The local coordinates η^{δ} of the point R are given by

$$(21)\quad \frac{\eta^{\delta}}{(g_{\delta\delta})^{1/2}} = d d_{\alpha\beta} u'^{\alpha} u'^{\beta} \frac{\delta^{12}_{\gamma\sigma} u'^{\gamma} d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma} d_{\mu\tau} \overline{\Gamma}^{\mu}_{\alpha\beta} u'^{\alpha} u'^{\beta} u'^{\gamma} + d_{\alpha\sigma} d_{\beta\tau} u'^{\alpha} u''^{\beta})},$$

where δ is not summed, and where $d \equiv |d_{\alpha\beta}|$ has the value $e(hg)^{1/2}$.

2. **Dual geodesics.** We consider next the line l_2 lying in the tangent plane to the surface at P which is in G. M. Green's "relation R"[7] to the normal l_1 to the surface at P. It can readily be shown that the equation of the line l_2 in local coordinates is

$$(22)\quad (g_{22})^{1/2} \Gamma^2_{12} \xi^1 + (g_{11})^{1/2} \Gamma^1_{12} \xi^2 + (g_{11} g_{22})^{1/2} = 0.$$

A geodesic curve on the surface has the property that its osculating plane at every point P contains the normal line l_1 to the surface at P. We may define a *dual geodesic* to be a curve on the surface which enjoys the property that its ray-point R corresponding to every point P lies on the line l_2 given by (22). If we require that the coordinates (21) satisfy the equation (22), we find, after some reduction, that the curve C is a solution of the differential equation

$$(23)\quad d\delta^{12}_{\alpha\beta} u'^{\alpha} u''^{\beta} + [d_{\alpha\beta}(d_{\gamma 1}\Gamma^1_{12} - d_{\gamma 2}\Gamma^2_{12}) + \delta^{\sigma\tau}_{12} d_{\gamma\sigma} d_{\mu\tau} \overline{\Gamma}^{\mu}_{\alpha\beta}] u'^{\alpha} u'^{\beta} u'^{\gamma} = 0,$$

where $\overline{\Gamma}^{\mu}_{\alpha\beta}$ is given by (6).

If we write $u^1 = u$, $u^2 = v$, $d_{11} = D$, $d_{12} = d_{21} = D'$, $d_{22} = D''$, $v' = dv/du$, $v'' = d^2v/du^2$, and make use of (6), equation (23) takes the form

$$(24)\quad (DD'' - D'^2)v'' = p + qv' + rv'^2 + sv'^3,$$

where

$$\begin{aligned}
p &= D'D_u - DD'_u + (2\Gamma^2_{12} - \Gamma^1_{11})DD' - \Gamma^2_{11} D'^2, \\
q &= D''D_u - DD''_u + D'D_v - DD'_v + (DD'' + D'^2)(2\Gamma^2_{12} - \Gamma^1_{11}) \\
&\quad + D^2\Gamma^1_{22} + DD'(\Gamma^2_{22} - 2\Gamma^1_{12}) - 2D'D''\Gamma^2_{11}, \\
r &= D''D_v - DD''_v + D''D'_u - D'D''_u \\
&\quad + (DD'' + D'^2)(\Gamma^2_{22} - 2\Gamma^1_{12}) - D''^2\Gamma^2_{11} \\
&\quad + D'D''(2\Gamma^2_{12} - \Gamma^1_{11}) + 2DD'\Gamma^1_{22}, \\
s &= D''D'_v - D'D''_v + (\Gamma^2_{22} - 2\Gamma^1_{12})D'D'' + \Gamma^1_{22} D'^2.
\end{aligned} \qquad (25)$$

[7] Lane, loc. cit., p. 81.

If the asymptotic lines are taken as parametric, we have $D = D'' = 0$, and equation (24) takes the form

$$v'' = \Gamma_{11}^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)v' + (2\Gamma_{12}^1 - \Gamma_{22}^2)v'^2 - \Gamma_{22}^1 v'^3. \tag{26}$$

The geodesics on a surface satisfy the differential equation[8]

$$v'' = -\Gamma_{11}^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)v' + (2\Gamma_{12}^1 - \Gamma_{22}^2)v'^2 + \Gamma_{22}^1 v'^3. \tag{27}$$

From (26) it is seen that if the asymptotic curve given by $v = \text{const.}$ is a dual geodesic, then $\Gamma_{11}^2 = 0$, which, by (27), is the condition that the curve be a geodesic. But if a curve is both an asymptotic and a geodesic it is a straight line. Hence, *if an asymptotic line is a dual geodesic it is a straight line.*

Equation (26) is independent of D'. Hence, we conclude that *isometric surfaces have the same equations of dual geodesics.*

It can be shown that when the asymptotics are parametric the directions of Segrè[9] at the point P on the surface are given by

$$\Gamma_{11}^2 du^3 - \Gamma_{22}^1 dv^3 = 0. \tag{28}$$

Comparison of equations (26) and (27) shows that the *directions of Segrè may be characterized as the directions in which the geodesics and dual geodesics coincide.*

3. **Ray-points of geodesics.** Each curve through the point P on the surface has a ray-point R. We shall now find the locus of the ray-points for all geodesics through P. Along a geodesic curve we have

$$u''^{\alpha} + \Gamma_{\beta\gamma}^{\alpha} u'^{\beta} u'^{\gamma} = 0. \tag{29}$$

In consequence of (29) and the definition of $\overline{\Gamma}_{\alpha\beta}^{\mu}$ from (6), the coordinates (η^1, η^2) of the ray-point R of a geodesic curve $u^\alpha = u^\alpha(s)$ through P are found from (21) to be given by

$$\begin{aligned} T\eta^1 &= -(g_{11})^{1/2}(u'^1)^2(u'^2), \\ T\eta^2 &= (g_{22})^{1/2}(u'^1)(u'^2)^2, \end{aligned} \tag{30}$$

where

$$T = \Gamma_{11}^2(u'^1)^3 + \Gamma_{12}^2(u'^1)^2(u'^2) - \Gamma_{12}^1(u'^1)(u'^2)^2 - \Gamma_{22}^1(u'^2)^3. \tag{31}$$

Elimination of u'^1 and u'^2 between the equations (30) yields a cubic equation which may be written in the form

[8] Eisenhart, *Differential Geometry*, Ginn and Co., 1909, p. 205.

[9] Lane, loc. cit., p. 77.

$$\Gamma_{11}^{2}\left(\frac{\overset{1}{\eta}}{(g_{11})^{1/2}}\right)^{3}+\Gamma_{22}^{1}\left(\frac{\overset{2}{\eta}}{(g_{22})^{1/2}}\right)^{3} - \frac{\overset{1}{\eta}\overset{2}{\eta}}{(g_{11}g_{22})^{1/2}}\left(\frac{\Gamma_{12}^{2}}{(g_{11})^{1/2}}\eta^{1}+\frac{\Gamma_{12}^{1}}{(g_{22})^{1/2}}\eta^{2}+1\right)=0. \tag{32}$$

From the form of equation (32) it can be seen that the locus has a double point at P, with the asymptotics at P as double point tangents. Furthermore, the curve has three points of inflection which lie on the line represented by equation (22), some one of the inflections lying on each of the lines through P given by the equation

$$\Gamma_{11}^{2}\left(\frac{\eta^{1}}{(g_{11})^{1/2}}\right)^{3}+\Gamma_{22}^{1}\left(\frac{\eta^{2}}{(g_{22})^{1/2}}\right)^{3}=0. \tag{33}$$

On comparing equations (28) and (33), we recognize that the latter gives the directions of Darboux[10] at P. Hence, we have the following theorem. *The locus of the ray-points of the geodesics through a point P on a surface is a cubic curve lying in the tangent plane to the surface at P. The asymptotics are the double point tangents to the curve at P, and the three points of inflection of the curve are given by the intersections of the tangents of Darboux at P with the line which is in Green's Relation R with the normal to the surface at the point P.* Lane[11] gives a similar theorem for projective space.

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[10] Lane, loc. cit., p. 76.

[11] Lane, loc. cit., p. 98.