A GENERALIZATION OF THE POLAR REPRESENTATION OF NONSINGULAR MATRICES

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1. **Introduction.** If A is a square matrix with elements in the complex number field, then

$$A = PU, \tag{1}$$

where P is a positive definite hermitian matrix and U is a unitary matrix.[1] In this polar representation of the matrix A, as it is called, the two matrices P and U are unique. Since the matrix P is positive definite and nonsingular, it has the same signature as the identity matrix E while the unitary matrix U is a conjunctive automorph of E. From (1) we may deduce a somewhat similar representation of A in terms of a positive definite hermitian matrix and a conjunctive automorph, not of E, but of any nonsingular positive definite hermitian matrix.

Let H be a nonsingular hermitian matrix which is positive definite, so that there exists a nonsingular matrix Q satisfying

$$Q^{-1}H(Q^{-1})^* = E,$$

where Q^* is the conjugate transposed of Q. If $B=Q^{-1}AQ$ and $B=PU$ is the polar representation of B, then $A=QPQ^{-1}QUQ^{-1}=DR$, where $D=QPQ^{-1}$ and $R=QUQ^{-1}$. Since $DH=QPQ^{-1}QQ^*=QPQ^*$, DH is hermitian with the same signature as H. Further $RHR^*=QUQ^{-1}QQ^* \cdot (Q^{-1})^*U^*Q^*=QQ^*=H$. Hence we have this result as an analogue of the polar representation (1) of A.

RESULT (1). *If H is any nonsingular positive definite hermitian matrix and A is a nonsingular matrix, then*

$$A = DR, \tag{2}$$

where DH is a positive definite hermitian matrix and $RHR^*=H$.

If $H=H^{-1}$, $A=DR=DHHR$ and, since $HRH(HR)^*=H^3=H$, $A=P_1R_1$, where P_1 is a positive definite hermitian matrix and $R_1HR_1^*=H$. Therefore we have as a second analogue of (1) the following result.

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[1] L. Autonne, *Sur les groupes linéaires, réels et orthogonaux*, Bulletin de la Société Mathématique de France, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

RESULT (2). *If H is a positive definite hermitian matrix, whose square is the identity matrix, and, if A is a nonsingular matrix, then*

$$A = P_1R_1, \tag{3}$$

where P_1 is a positive definite hermitian matrix and R_1 is a conjunctive automorph of H. The two matrices P_1 and R_1 are unique.

It is our intention here to determine what corresponds to results (1) and (2), if H is a nonsingular hermitian matrix but not necessarily definite.

2. **Generalization in the complex field.** Let H be any nonsingular hermitian matrix and let A be a nonsingular matrix. If

$$A = DR, \tag{4}$$

where

$$DH = HD^* \tag{5}$$

and

$$RHR^* = H, \tag{6}$$

we shall call (4) *a polar representation of A with respect to H* or, for brevity an *H-representation* of A. If (4), (5) and (6) are satisfied, $AHA^*H^{-1} = DRHR^*D^*H^{-1} = DHD^*H^{-1} = D^2$. Further, if $AHA^*H^{-1} = D^2$ and $DH = HD^*$, then (6) is satisfied with $R = D^{-1}A$. For $D^{-1}AH(D^{-1}A)^* = D^{-1}AHA^*(D^{-1})^* = D^{-1}D^2H(D^{-1})^* = H$ by (5). Therefore we have proved this lemma.

LEMMA 1. *A necessary and sufficient condition that a matrix A have an H-representation is that there exist a matrix D such that $DH = HD^*$ and $AHA^*H^{-1} = D^2$.*

If A has the H-representation (4) and $QAQ^{-1} = A_1$, $QDQ^{-1} = D_1$, $QRQ^{-1} = R_1$ and $QHQ^* = H_1$, then $A_1 = D_1R_1$ where $D_1H_1 = H_1D_1^*$ and $R_1H_1R_1^* = H_1$. Moreover $H_1 = QHQ^*$ and $A_1H_1A_1^* = QAHA^*Q^*$, so that the two pencils of hermitian matrices $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive. Therefore, if A has an H-representation and if the two pencils $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive, the matrix A_1 has an H_1-representation. Accordingly we may suppose that the pencil $AHA^* - xH$ is in a canonical or normal form previously determined.[2] Since DH is hermitian, the elementary di-

[2] G. R. Trott, *On the canonical form of a nonsingular pencil of hermitian matrices*, American Journal of Mathematics, vol. 56 (1934), pp. 359–371. H. W. Turnbull, *Pencils of hermitian forms*, Proceedings of the London Mathematical Society (2), vol. 39 (1935), pp. 232–248.

visors of $D-xE$ are the same as those of the pencil of hermitian matrices $DH-xH$. Complex elementary divisors of such a pencil must therefore occur in conjugate pairs and in particular this is true of the pure imaginary elementary divisors. Hence, if $D-xE$ has the elementary divisor $(x-ib)^r$ repeated s times, where b is real, then $D-xE$ also has the elementary divisor $(x+ib)^r$ repeated s times and D^2-xE has the elementary divisor $(x+b^2)^r$ repeated $2s$ times. Since the elementary divisors of D^2-xE are the same as those of AHA^*-xH, the matrix A cannot have an H-representation if the pencil AHA^*-xH has an elementary divisor $(x+b^2)^r$, that is, a negative elementary divisor, repeated an odd number of times.

We proceed to determine what further conditions, if any, must be satisfied in order that A may have an H-representation. In canonical form the matrices H and AHA^* are similarly partitioned diagonal block matrices of such a nature that it is only necessary to consider the three special cases in which the pencil AHA^*-xH has

(i) only the two elementary divisors $(x-a)^n$, $(x-\bar{a})^n$, $a\neq\bar{a}$,

(ii) the single elementary divisor $(x-b^2)^n$, b real,

(iii) only the two elementary divisors $(x+b^2)^n$, $(x+b^2)^n$, b real.

Case (i). In canonical form

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \quad \text{and} \quad AHA^*H^{-1} = \begin{pmatrix} a(E+U) & 0 \\ 0 & \bar{a}(E+U)' \end{pmatrix},$$

where U is the auxiliary unit matrix of order n. Let

$$G = (E+U)^{1/2} = E + U/2 - U^2/8 + \cdots + \alpha U^{n-1}, \tag{7}$$

$$2^{2n-4}(n-3)!(n-1)!\alpha = (-1)^n(2n-5)!.$$

Then $G' = (E+U')^{1/2}$. If $a=\rho^2 e^{i\theta}$, $a^{1/2}=\delta\rho e^{i\theta/2}$ and $\bar{a}^{1/2}=\delta\rho e^{-i\theta/2}$, where $\delta=\pm 1$. Therefore, if $D=[a^{1/2}G, \bar{a}^{1/2}G']$, $DH=HD^*$ and $D^2=AHA^*H^{-1}$. The signature of DH is the same as that of H since the signature of both matrices is zero. Since δ may have either of the values 1 or -1, D is not unique.

Case (ii). In canonical form

$$H = \epsilon T \quad \text{and} \quad AHA^* = b^2(E+U)H,$$

where $\epsilon=\pm 1$ and T is the counter unit matrix, so that

$$T^2 = E \quad \text{and} \quad TU = U'T. \tag{8}$$

If $D=bG$, where G is defined by (7), $D^2=AHA^*H^{-1}$ and $DH=HD^*$ by (8). If n is even the signatures of DH and H are both zero and, if in DH, b is replaced by $-b$ the signature of the resulting matrix is also zero. If n is odd, the signature of H is ϵ and that of DH is $\pm\epsilon$ according as b is positive or negative. Therefore, if DH has the same signature as H and n is even, D is not unique but, if n is odd, D is unique.

Case (iii). If the matrix D exists the pencil $DH-xH$ has only the two elementary divisors $(x+ib)^n$, $(x-ib)^n$ and we may take $DH-xH$ in the canonical form

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} ibG & 0 \\ 0 & -ibG' \end{pmatrix}, \tag{9}$$

where G is defined by (7). Hence

$$AHA^*H^{-1} = D^2 = \begin{pmatrix} -b^2(E+U) & 0 \\ 0 & -b^2(E+U') \end{pmatrix}$$

and

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

while the matrix H may be transformed without disturbing D^2 into

$$\begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}.$$

For a matrix pencil AHA^*-xH with elementary divisors $(x+b^2)^n$, $(x+b^2)^n$ there are three distinct possible canonical forms in which H is one of the matrices

$$\begin{pmatrix} \epsilon_1 T & 0 \\ 0 & \epsilon_2 T \end{pmatrix},$$

$\epsilon_1=1$, $\epsilon_2=-1$; $\epsilon_1=\epsilon_2=1$ or $\epsilon_1=\epsilon_2=-1$. If we call ϵ_1 and ϵ_2 the *indices* associated with the elementary divisor $(x+b^2)^n$, we see that A has an H-representation, if and only if one index is positive and the other is negative. Further, even when A does have an H-representation, D is not unique. Since the canonical form for the pencil AHA^*-xH is diagonal block, the following theorem follows immediately from Lemma 1.

THEOREM 1. *Let H be any nonsingular hermitian matrix and A be a nonsingular matrix. Then A has an H-representation $A=DR$, if and only if the negative elementary divisors of the pencil AHA^*-xH occur in pairs and exactly half of the indices associated with each negative elementary divisor are positive. The matrix, D, and therefore the matrix*

R, is unique, if and only if all elementary divisors of the pencil are positive and of odd order.

If $H^2=E$, $A=DHHR=SV$, where $S=DH$ and $V=HR$. Moreover $S=S^*$ and $VHV^*=H^3=H$. Hence we have as a corollary the following.

COROLLARY 1. *If $H^2=E$, the matrix A can be written in the form $A=SV$, where S is hermitian with the same signature as H and, $VHV^*=H$, if and only if the conditions of Theorem 1 are satisfied.*

The known theorem on the uniqueness of the polar components of the matrix A in (1) is a particular case of Theorem 1. For, if $H=E$, $AHA^*=AA^*$ and the elementary divisors of the pencil AA^*-xE are all positive and linear.

3. **Modified representation of any nonsingular matrix.** Even if the matrix A does not satisfy the conditions of Theorem 1, it is possible to find a somewhat different representation of A. We first reduce the pencil AHA^*-xH by conjunctive transformation to

$$\begin{pmatrix} A_1H_1A_1^* & 0 \\ 0 & A_2H_2A_2^* \end{pmatrix} - x\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where no elementary divisor of $A_1H_1A_1-xH_1$ is negative and all elementary divisors of $A_2H_2A_2^*-xH_2$ are negative. Then $A_1=D_1R_1$ and, since all elementary divisors of $-A_2H_2A_2^*-xH_2$ are positive, there exists a matrix D_2, such that $D_2^2=-A_2H_2A_2^*H_2^{-1}$ and $D_2H_2=H_2D_2^*$. If $R_2=D_2^{-1}A_2$, then $R_2H_2R_2^*=D_2^{-1}A_2H_2A_2^*(D_2^*)^{-1}=D_2\cdot(-D_2^2H_2)(D_2^*)^{-1}=-H_2$. Therefore $A_2=D_2R_2$, where D_2H_2 is hermitian with the same signature as H_2 and $R_2H_2R_2^*=-H_2$. Let

$$V=\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix} = [E_1, -E_2],$$

where E_i is the unit matrix of the same order as H_i. Then $V^2=E$. If $A=[A_1, A_2]$, $H=[H_1, H_2]$, $R=[R_1, R_2]$ and $D=[D_1, D_2]$, then $A=DR$ where DH is hermitian with the same signature as H and $RHR^*=HV$, so that the signature of HV is the same as that of H. Further from its form it is apparent that V is a polynomial in AHA^*H^{-1} and that $VH=HV=HV^*$, so that $VHV^*=H$. While we have used a special form of the pencil AHA^*-xH it follows that similar results are true when the pencil is not in this form. Accordingly we have proved this theorem.

THEOREM 2. *Let H be a nonsingular hermitian matrix and let A be* *a nonsingular matrix of the same order as H. Then there exists at least one matrix V, a polynomial in AHA^*H^{-1}, with the properties that $V^2=E$, VH is hermitian with the same signature as H and that no characteristic number of $AHA^*(VH)^{-1}$ is negative. The matrix $A=DR$ where DH is hermitian with the same signature as H and $RHR^*=VH$. If all the characteristic numbers of AHA^*H^{-1} are real, the matrix V is uniquely determined.*

4. **Representation in the real field.** If H is a real symmetric matrix and A a real nonsingular matrix our argument may be carried through in the field of real numbers with only a few alterations. In (i), if the pair of conjugate elementary divisors is $(x-c\pm id)^n$, we take $H=qT$, $AHA^*=p(E+U)qT$, where

$$p=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad \text{and} \quad q=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, if

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}^2 = p,$$

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix} GH$$

is symmetric and

$$\left\{\begin{pmatrix} r & s \\ -s & r \end{pmatrix} G\right\}^2 = AHA^*H^{-1}.$$

Case (ii) is unaltered. In (iii) we replace the matrices D and H of (9) by

$$\begin{pmatrix} 0 & bG \\ -bG & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix},$$

respectively. It therefore follows that Theorems 1 and 2 are true in the field of real numbers, if hermitian is replaced by symmetric and conjugate transposed by transposed.

5. **Conditions for the commutativity of the polar components.** Let the matrix A have an H-representation $A=DR$. Then $A=RD_1$ where $D_1=R^{-1}DR$. The matrix D_1H is hermitian. Further $HA^*H^{-1}A=HD_1^*R^*H^{-1}RD_1=D_1HH^{-1}D_1=D_1^2$. Since $D^2=AHA^*H^{-1}$, it follows that, if A^* is commutative with $H^{-1}AH$, $D_1^2=D^2$ and that for a proper choice $D_1=D$. Conversely, if $D_1=D$, $H^{-1}AH$ is commutative with A^*. We have therefore proved this theorem.

THEOREM 3. *If A has an H-representation $A=DR$, A also has an H-representation $A=RD_1$. A necessary and sufficient condition that $D=D_1$ or that D be commutative with R is that A^* be commutative with $H^{-1}AH$.*

Therefore, if $A=DR$ and A is normal[3] with respect to H, $A=RD$ so that R and D are commutative. That the converse is not true may be shown as follows. Let

$$H=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad\text{and}\quad A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

Then

$$AHA^*H^{-1}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\1&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&0\\0&1&0&0\end{pmatrix}=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

The characteristic numbers of AHA^*H^{-1} are all plus one and therefore A has an H-representation. In fact

$$D=\begin{pmatrix}1&1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix}\quad\text{and}\quad R=\begin{pmatrix}1&-1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix}.$$

Hence

$$H^{-1}AH=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and is commutative with A^*. However, $H^{-1}AH$ is not a polynomial in A^* and is therefore not normal with respect to H.

If the elementary divisors of AHA^*-xH are all positive and linear and if $A=DR=RD$, it is comparatively easy to show that $H^{-1}AH$ is a polynomial in A^* and therefore that A is normal with respect to H.

[3] John Williamson, *Matrices normal with respect to an hermitian matrix*, American Journal of Mathematics, vol. 60 (1938), pp. 355–373.

This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix A to be commutative is that A be a normal matrix.

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REMARKS ON REGULARITY OF METHODS OF SUMMATION

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A doubly infinite matrix[1] (a_{mn}) $(m, n=1, 2, \cdots)$ is said to be *regular*, if for every sequence $x=\{x_n\}$ with limit x' the corresponding sums $y_m=\sum_{n=1}^{\infty}a_{mn}x_n$ exist for $m=1, 2, \cdots$, and if $\lim_{m\to\infty} y_m=x'$. An apparently more inclusive definition of regularity is that for each sequence x with limit x' the sums defining y_m shall exist for all $m\geqq m_0(x)$ and $\lim_{m\to\infty} y_m=x'$. Tamarkin[2] has shown that (a_{mn}) is regular in the latter sense if and only if there exists an m_1 *independent of* x such that the matrix (a_{mn}) $(m\geqq m_1,\ n\geqq 1)$ is regular in the former sense. Using point set theory in the Banach space (c), he proves a theorem[3] from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

THEOREM 1. *Suppose the doubly infinite matrix* (a_{mn}) *has the property that for each sequence* $x=\{x_n\}$ *with limit* 0 *there exists an* $m_0=m_0(x)$ *such that for all* $m\geqq m_0(x)$, $u_m=\lim\ \sup_{k\to\infty}\left|\sum_{n=1}^{k}a_{mn}x_n\right|<\infty$. *Then there exists an* m_1 *such that* $\sum_{n=1}^{\infty}|a_{mn}|<\infty$ *for all* $m\geqq m_1$.

If in addition $\lim_{m\to\infty} u_m=0$ for each sequence x with limit 0, it will follow[4] that there exists an N such that $\sum_{n=1}^{\infty}|a_{mn}|\leqq N<\infty$, for all $m\geqq m_1$.

To prove Theorem 1, suppose there were an infinite sequence $m_1<m_2<\cdots$ such that $\sum_{n=1}^{\infty}|a_{mn}|=\infty$ for $m\in\{m_\nu\}$. Let $x_1, \cdots, x_{k_1}$ be chosen with unit moduli and with amplitudes such that

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[1] In this note a_{mn}, x_n and x' denote finite complex numbers.

[2] J. D. Tamarkin, *On the notion of regularity of methods of summation of infinite series*, this Bulletin, vol. 41 (1935), pp. 241–243.

[3] J. D. Tamarkin, loc. cit., p. 242, lines 1–6.

[4] See, for example, I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, vol. 151 (1921), pp. 79–111; p. 85, Theorem 4.