position fields coincide and are cyclic. The field $\bar{L}$ is then equivalent to a subfield of $\bar{K}'$; without loss of generality we may suppose $\bar{K}' > \bar{L} \geqq \bar{K}$. The degree $[\bar{L}:\bar{K}] = \bar{m}$ is a divisor of m. Consequently $[Z_n\bar{L}:\bar{K}] = [Z_n\bar{L}:\bar{L}][\bar{L}:\bar{K}] = n\bar{m}$. By the Galois theory there is then for every integer n an extension Z_n^* of degree n over $\bar{K}$. The defining equation $f^*(x) = 0$ of $Z_n^*/\bar{K}$ now may be approximated by an irreducible equation $f(x) = 0$ of degree n with coefficients in K so that Z_n^* is generated by the roots of $f(x) = 0$. The root field of $f(x) = 0$ over K is the cyclic extension Z_n' of degree n over K. Hence $Z_n^* = Z_n'\bar{K}$ for all n, contrary to the assumption that K is not relatively complete with respect to any rank one valuation.

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A DIFFERENTIAL GEOMETRY PROBLEM USING TENSOR ANALYSIS

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1. **Introduction.** The problem at hand was worked out in attempting to apply tensors to a much more general problem in classical differential geometry. The results obtained in a general coordinate system reduce readily to classical results of Eisenhart. An interesting interpretation of Christoffel symbols appears.

2. *R* **net.** A rectilinear congruence in 3-space is called a W-congruence if the asymptotic lines on the two focal surfaces correspond. If the tangents to both families of curves of a conjugate net on a surface form W-congruences the net is called an R net.[1] We derive the analytic conditions that must obtain in order that a given conjugate net on a surface shall be an R net.

3. **Equations for an** *R* **net.** Let S_1 be one focal surface of a W-congruence, the vector equation of the surface being

$$\overset{\alpha}{z}_1 = \overset{\alpha}{z}_1(\overset{i}{x}), \qquad \alpha = 1, 2, 3; \; i = 1, 2. \tag{3.1}$$

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[1] Tzitzeica, Comptes Rendus de l'Académie des Sciences, Paris, vol. 152 (1911), p. 1077.

Let S_2 be the other focal surface of the congruence with vector z_2^{α} so that we have

$$z_2^{\alpha} = z_1^{\alpha} + \rho_1 \xi_1^{\alpha} \tag{3.2}$$

where ξ_1^{α} is a unit vector tangent to S_1, and ρ_1 is an invariant.

Then if $\lambda_{1/}^{i}$ are the components of this vector in the x's we have

$$\xi_1^{\alpha} = z_{1/,i}^{\alpha} \lambda_{1/}^{i} \tag{3.3}$$

where in this case

$$z_{1/,i}^{\alpha} = \frac{\partial z_1^{\alpha}}{\partial x^i}$$

since z_1^{α} being an invariant for a transformation of coordinates in the x's, the ordinary derivative of z_1^{α} with respect to x^i is the same as the covariant derivative with respect to g_{ij}, the fundamental tensor of S_1. Then substituting (3.3) in (3.2) we have

$$z_2^{\alpha} = z_1^{\alpha} + \rho_1 z_{1/,i}^{\alpha} \lambda_{1/}^{i}. \tag{3.4}$$

Similarly by the property of focal surfaces we have

$$z_1^{\alpha} = z_2^{\alpha} + \rho_2 z_{2/,i}^{\alpha} \lambda_{2/}^{i} \tag{3.5}$$

where ρ_2 is an invariant, and $\lambda_{2/}^{i}$ is a unit vector tangent to S_2.

Adding (3.4) and (3.5) we have

$$\rho_1 \lambda_{1/}^{i} z_{1/,i}^{\alpha} + \rho_2 \lambda_{2/}^{i} z_{2/,i}^{\alpha} = 0. \tag{3.6}$$

From (3.6) we have

$$\rho_1 = \bar{e}\rho_2, \qquad \lambda_{1/}^{i} z_{1/,i}^{\alpha} = e \lambda_{2/}^{i} z_{2/,i}^{\alpha} \tag{3.6$'$}$$

where $\bar{e}=1$ if $e=-1$, and $\bar{e}=-1$ if $e=1$, and conversely.

We differentiate (3.4) covariantly and have

$$z_{2/,k}^{\alpha} = z_{1/,k}^{\alpha} + \rho_{1/,k} \lambda_{1/}^{i} z_{1/,i}^{\alpha} + \rho_1 \lambda_{1/,k}^{i} z_{1/,i}^{\alpha} + \rho_1 \lambda_{1/}^{i} z_{1/,ik}^{\alpha}. \tag{3.7}$$

Let η_1^{α} be the unit normal to S_1.

We multiply (3.7) by $\lambda_{2/}^{k}$, sum for k, multiply by η_1^{α}, sum for α and we have

$$0 = \rho_1 b_{1/ij} \lambda_{1/}^{i} \lambda_{2/}^{j} \tag{3.8}$$

the first three terms on the right vanishing because η_1^{α} is perpendicular to S_1, the term on the left vanishing because of this fact and (3.6).

and the last term becoming $\rho_1\eta_1^\alpha \cdot z^\alpha_{i/,ij}\lambda^i_{1/}\lambda^j_{2/}$ the second factor of which is denoted by $b_{1/ij}\lambda^i_{1/}\lambda^j_{2/}$.[2]

Hence *the directions* $\lambda^i_{1/}$ *and* $\lambda^j_{2/}$ *are conjugate on* S_1.[3]

Similarly they are conjugate on S_2.

We next differentiate (3.7) covariantly with respect to g_{ij}, the fundamental tensor of S_1,

$$
\begin{aligned}
\frac{\partial^2 \overset{\alpha}{z}_2}{\partial x^k \partial x^j} - \overset{\alpha}{z}_{2/,m}\left\{ {m \atop kj} \right\}_{g_{ij}} &= \overset{\alpha}{z}_{1/,kj} + \rho_{1/,kj}\overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,i} + \rho_{1/,k}\overset{i}{\lambda}_{1/,j}\overset{\alpha}{z}_{1/,i} \\
&+ \rho_{1/,k}\overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,ij} + \rho_{1/,j}\overset{i}{\lambda}_{1/,k}\overset{\alpha}{z}_{1/,i} + \rho_1\overset{i}{\lambda}_{1/,kj}\overset{\alpha}{z}_{1/,i} + \rho_1\overset{i}{\lambda}_{1/,k}\overset{\alpha}{z}_{1/,ij} \\
&+ \rho_{1/,j}\overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,ik} + \rho_1\overset{i}{\lambda}_{1/,j}\overset{\alpha}{z}_{1/,ik} + \rho_1\overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,ikj}.
\end{aligned} \tag{3.9}
$$

Multiply by η_2^α (the unit normal to S_2), sum for α and we have

$$
\begin{aligned}
b_{2/kj} &= b_{1/kj}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 + \rho_{1/,kj}\overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,i}\cdot\overset{\alpha}{\eta}_2 + \rho_{1/,k}\overset{i}{\lambda}_{1/,j}\overset{\alpha}{z}_{1/,i}\cdot\overset{\alpha}{\eta}_2 \\
&+ \rho_{1/,k}\overset{i}{\lambda}_{1/}b_{1/ij}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 + \rho_{1/,j}\overset{i}{\lambda}_{1/,k}\overset{\alpha}{z}_{1/,i}\cdot\overset{\alpha}{\eta}_2 + \rho_1\overset{i}{\lambda}_{1/,kj}\overset{\alpha}{z}_{1/,i}\cdot\overset{\alpha}{\eta}_2 \\
&+ \rho_1\overset{i}{\lambda}_{1/,k}b_{1/ij}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 + \rho_{1/,j}\overset{i}{\lambda}_{1/}b_{1/ik}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 + \rho_1\overset{i}{\lambda}_{1/,j}b_{1/ik}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 \\
&+ \rho_1\overset{i}{\lambda}_{1/}b_{1/ik,j}\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2 + b_{1/ik}\rho_1\overset{i}{\lambda}_{1/}\overset{\alpha}{\eta}_{1/,j}\cdot\overset{\alpha}{\eta}_2.
\end{aligned} \tag{3.10}
$$

In the future, since unless otherwise stated covariant differentiation is with respect to the fundamental tensor of S_1, we shall note the covariant derivative of $z^\alpha_{2/,k}$ by $z^\alpha_{2/,kj}$.

We evaluate $z^\alpha_{1/,i}\cdot\eta_2^\alpha$ as follows.

We differentiate (3.5) covariantly giving

$$
\overset{\alpha}{z}_{1/,i} = \overset{\alpha}{z}_{2/,i} + \rho_{2/,i}\overset{s}{\lambda}_{2/}\overset{\alpha}{z}_{2/,s} + \rho_2\overset{s}{\lambda}_{2/,i}\overset{\alpha}{z}_{2/,s} + \rho_2\overset{s}{\lambda}_{2/}\overset{\alpha}{z}_{2/,si}. \tag{3.11}
$$

Multiply by η_2^α and sum for α, giving

$$
\overset{\alpha}{z}_{1/,i}\cdot\overset{\alpha}{\eta}_2 = \rho_2\overset{s}{\lambda}_{2/}b_{2/si}. \tag{3.12}
$$

Substituting this value for $z^\alpha_{1/,i}\cdot\eta_2^\alpha$ in (3.10) we have

$$
\begin{aligned}
b_{2/kj} &= (\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2)(b_{1/kj} + b_{1/ij}\rho_{1/,k}\overset{i}{\lambda}_{1/} + b_{1/ij}\rho_1\overset{i}{\lambda}_{1/,k} + b_{1/ik}\rho_{1/,j}\overset{i}{\lambda}_{1/} \\
&+ b_{1/ik}\rho_1\overset{i}{\lambda}_{1/,j} + b_{1/ik,j}\rho_1\overset{i}{\lambda}_{1/}) + \rho_1\overset{i}{\lambda}_{1/}b_{1/ik}\overset{\alpha}{\eta}_{1/,j}\cdot\overset{\alpha}{\eta}_2 \\
&+ \rho_{1/,k}\overset{i}{\lambda}_{1/,j}\rho_2\overset{s}{\lambda}_{2/}b_{2/si} + \rho_{1/,j}\overset{i}{\lambda}_{1/,k}\rho_2\overset{s}{\lambda}_{2/}b_{2/si} \\
&+ \rho_1\overset{i}{\lambda}_{1/,kj}\rho_2\overset{s}{\lambda}_{2/}b_{2/si}.
\end{aligned} \tag{3.13}
$$

Since the asymptotic lines on S_1 and S_2 are to correspond we have

[2] L. P. Eisenhart, *Riemannian Geometry*, 1926, Equation 56.2, p. 189.

[3] Ibid., Equation 56.3, p. 189.

$$b_{2/is} = \mu b_{1/is}. \tag{3.14}$$

We determine μ as follows. Differentiate the second of (3.6′) covariantly, giving

$$\lambda^i_{1/,j} z^\alpha_{1/,i} + \lambda^i_{1/} z^\alpha_{1/,ij} + e(\lambda^i_{2/,j} z^\alpha_{2/,i} + \lambda^i_{2/} z^\alpha_{2/,ij}) = 0. \tag{3.15}$$

Multiply by η^α_2, sum for α and use (3.12). This becomes

$$\lambda^i_{1/} b_{1/ij}(\eta^\alpha_1 \cdot \eta^\alpha_2) + \rho_2 \lambda^i_{1/,j} \lambda^k_{2/} b_{2/ki} + e\lambda^i_{2/} b_{2/ij} = 0. \tag{3.16}$$

Now multiply by $\lambda^j_{1/}$ and sum for j giving

$$(\eta^\alpha_1 \cdot \eta^\alpha_2) b_{1/ij} \lambda^i_{1/} \lambda^j_{1/} + b_{2/ik} \rho_2 \lambda^i_{1/,j} \lambda^k_{2/} \lambda^j_{1/} = 0. \tag{3.17}$$

But since $b_{2/ik} = \mu b_{1/ik}$ we have

$$\mu = - \frac{(\eta^\alpha_1 \cdot \eta^\alpha_2) b_{1/ij} \lambda^i_{1/} \lambda^j_{1/}}{\rho_2 b_{1/ik} \lambda^i_{1/,j} \lambda^k_{2/} \lambda^j_{1/}}. \tag{3.18}$$

Substituting in (3.13) and using the fact that[3]

$$\eta^\alpha_{1/,j} = - b_{1/ej} g^{em} z^\alpha_{1/,m}$$

we have

$$\begin{aligned} &\rho_2 b_{is} \lambda^s_{2/} \lambda^i_{1/,h} \lambda^h_{1/} [b_{kj} + \rho_{1/,k} \lambda^i_{1/} b_{ij} + \rho_1 \lambda^i_{1/,k} b_{ij} + \rho_{1/,j} \lambda^i_{1/} b_{ik} \\ &+ \rho_1 \lambda^i_{1/,j} b_{ik} + \rho_1 \lambda^i_{1/} b_{ik,j}] + b_{hr} \lambda^h_{1/} \lambda^r_{1/} [b_{kj} - \rho_{1/,k} \lambda^i_{1/,j} \rho_2 \lambda^s_{2/} b_{is} \\ &- \rho_1 \rho_2 \lambda^i_{1/,kj} \lambda^s_{2/} b_{is} + b_{ik} \rho_1 \rho_2 \lambda^i_{1/} b_{ej} g^{em} b_{ms} \lambda^s_{2/} \\ &- \rho_{1/,j} \lambda^i_{1/,k} \rho_2 \lambda^s_{2/} b_{is}] = 0 \end{aligned} \tag{3.19}$$

where the b's are all those of S_1.

To evaluate ρ_1, multiply (3.16) by $\lambda^j_{2/}$ and sum for j, giving, by (3.14)

$$\rho_1 b_{1/ki} \lambda^k_{2/} \lambda^i_{1/,j} \lambda^j_{2/} + \lambda^i_{2/} \lambda^j_{2/} b_{1/ij} = 0. \tag{3.20}$$

Similarly, we have

$$\rho_2 b_{1/ki} \lambda^k_{1/} \lambda^i_{2/,j} \lambda^j_{1/} + \lambda^i_{1/} \lambda^j_{1/} b_{1/ij} = 0 \tag{3.21}$$

where $\lambda^i_{2/,j}$ is with respect to $\bar{g}_{ij}$ of S_2. (It should be remarked that ρ_2 may be expressed entirely in terms of elements of S_1 by means by (3.6) and (3.7) and differentiation.)

Equations (3.19) with ρ_1 and ρ_2 determined by (3.20) and (3.21), respectively, constitute the condition that must obtain in order for the tangents to the curves of direction $\lambda^i_{1/}$ on S_1 to form a W-congruence. An equation similar to (3.19) obtains for the direction $\lambda^j_{2/}$. These two equations must hold in order for the net with directions $\lambda^i_{1/}$ and $\lambda^j_{2/}$ to be an R net.

In particular we consider the case where $\lambda^i_{1/}$ and $\lambda^j_{2/}$ are tangent to the u and v parametric curves, respectively. Then[4]

$$\begin{aligned} &\overset{1}{\lambda}_1 = 1, && b_{11} = D, \\ &\overset{2}{\lambda}_1 = 0, && b_{12} = b_{21} = D' = 0, \\ &\overset{1}{\lambda}_2 = 0, && b_{22} = D'', \\ &\overset{2}{\lambda}_2 = 1, && \end{aligned} \tag{3.22}$$

and it is easily shown that

$$\overset{i}{\lambda}_{a/,j} = \begin{Bmatrix} i \\ aj \end{Bmatrix}$$

for any *fixed* i, a, j, which is an interesting interpretation of the Christoffel symbols in this case.

In this case (3.19) reduces to

$$2\frac{\partial}{\partial v}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial u}\left(\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} - \frac{D''}{D}\begin{Bmatrix} 2 \\ 11 \end{Bmatrix}\right), \tag{3.23}$$

the equation obtained by Eisenhart.[5]

The equation similar to (3.19) reduces to

$$2\frac{\partial}{\partial u}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial v}\left(\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \frac{D}{D''}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}\right) \tag{3.24}$$

and these two equations constitute the condition that the parametric curves of a surface S form an R net.

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[4] L. P. Eisenhart, *Differential Geometry*, 1909, p. 115.

[5] L. P. Eisenhart, *Transformation of Surfaces*, 1923, p. 106.