ON THE LEAST SOLUTION OF PELL'S EQUATION

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Let x_0, y_0 be the least positive solution of Pell's equation

$$x^2 - dy^2 = 4,$$

where d is a positive integer, not a square, congruent to 0 or 1 (mod 4). Let $\epsilon = (x_0 + d^{1/2} y_0)/2$. It was proved by Schur[1] that

$$\epsilon < d^{d^{1/2}}, \tag{1}$$

or, more precisely,

$$\log \epsilon < d^{1/2}((1/2) \log d + (1/2) \log \log d + 1) . \tag{2}$$

He deduced (1) from (2) by the property that

$$d^{1/2}((1/2) \log d + (1/2) \log \log d + 1) < d^{1/2} \log d$$

for $d > 244.69 \cdots$, and, for $d \leqq 244$, (1) is established by direct computation. It is the object of the present note to establish a slightly better result that

$$\log \epsilon < d^{1/2}((1/2) \log d + 1). \tag{3}$$

Thus (1) follows immediately without any calculation. The method used is that described in the preceding paper.

Let $(d \mid r)$ be Kronecker's symbol. (We extend the definition to include negative values of r by the relation $(d \mid r_1) = (d \mid r_2)$ for $r_1 \equiv r_2 \pmod{d}$.)

Let f denote the fundamental discriminant related to d, that is,

$$d = m^2 f,$$

where f is not divisible by a square of odd prime and is either odd, or congruent to 8 or congruent to 12 (mod 16).

LEMMA 1. *For $d > 0$, we have*

$$\left(\frac{d}{r}\right) = \left(\frac{d}{-r}\right).$$

PROOF. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Theorem 101.

LEMMA 2. *We have*

Received by the editors December 3, 1941.

[1] Göttingen Nachrichter, 1918, pp. 30–36.

$$\sum_r \left(\frac{f}{r}\right) e^{2\pi i n r/f} = \left(\frac{f}{n}\right) f^{1/2},$$

where r runs over a complete residue system, mod f.

PROOF. Landau, loc. cit., Theorem 215.

LEMMA 3. *We have*

$$\frac{1}{A^*+1}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| \leqq \frac{1}{2}\left(f^{1/2} - \frac{A^*+1}{f^{1/2}}\right),$$

where A^* is the least positive residue of A, mod f.

PROOF. (See Lemma 1 of the preceding paper.) We have, by Lemma 2,

$$\begin{aligned} f^{1/2}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right) &= \frac{1}{2} f^{1/2}\sum_{a=0}^{A}\sum_{n=-a}^{a}\left(\frac{f}{n}\right) \\ &= \frac{1}{2}\sum_{a=0}^{A}\sum_{n=-a}^{a}\sum_{r=1}^{f}\left(\frac{f}{r}\right) e^{2\pi i n r/f} \\ &= \frac{1}{2}\sum_{r=1}^{f}\left(\frac{f}{r}\right)\sum_{a=0}^{A}\sum_{n=-a}^{a} e^{2\pi i n r/f}. \end{aligned}$$

Then

$$\begin{aligned} f^{1/2}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r=1}^{f-1}\left|\sum_{a=0}^{A}\sum_{n=-a}^{a} e^{2\pi i n r/f}\right| \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin (A+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin (A^*+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\sum_{a=0}^{A*}\sum_{n=-a}^{a} e^{2\pi i n r/f} \\ &= \frac{1}{2}((A^*+1)f - (A^*+1)^2), \end{aligned}$$

since

$$\sum_{r=1}^{f-1} e^{2\pi i n r/f} = \sum_{r=1}^{f} e^{2\pi i n r/f} - 1 = \begin{cases} -1 & \text{if } f \nmid n, \\ f-1 & \text{if } f \mid n. \end{cases}$$

LEMMA 4. *For any discriminant $d>0$ and $A>d^{1/2}$, we have*

$$\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| \leqq \frac{1}{2}Ad^{1/2}.$$

PROOF. It is well known that[2]

$$\left(\frac{d}{n}\right) = \left(\frac{f}{n}\right)\sum_{r\mid(m,n)}\mu(r).$$

Then

$$\begin{aligned}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right) &= \sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\sum_{r\mid(m,n)}\mu(r)\\ &= \sum_{r\mid m}\mu(r)\sum_{a=1}^{A}\sum_{n=1,\,r\mid n}^{a}\left(\frac{f}{n}\right) = \sum_{r\mid m}\mu(r)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{rn}\right)\\ &= \sum_{r\mid m}\mu(r)\left(\frac{f}{r}\right)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right).\end{aligned}$$

Then, by Lemma 2,

$$\begin{aligned}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r\mid m}\left|\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\left|\sum_{b=1}^{[A/r]}\sum_{n=1}^{b}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\left(\left(\left[\frac{A}{r}\right]+1\right)f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^2\right)\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\cdot\frac{A}{r}f^{1/2} \leqq \frac{1}{2}Af^{1/2}m = \frac{1}{2}Ad^{1/2},\end{aligned}$$

since we have $f^{1/2}r < f^{1/2}m < A$,

$$f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^2 < f^{1/2} - \frac{1}{f^{1/2}}\cdot f = 0$$

and

$$\sum_{r\mid m}1 \leqq m.$$

LEMMA 5. *We have*

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n} < \frac{1}{2}\log d + 1.$$

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[2] This follows from the fact that $\sum_{d\mid a}\mu(d) = 0$ or 1 according as $a > 1$ or $a = 1$.

PROOF. For $n \geqq 1$ let

$$S(n) = \sum_{a=1}^{n} \sum_{m=1}^{a} \left(\frac{d}{m}\right),$$

and let $S(0) = S(-1) = 0$. Then we have

$$S(n) - 2S(n-1) + S(n-2) = \left(\frac{d}{n}\right), \qquad n \geqq 1,$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{d}{n}\right)\frac{1}{n} &= \sum_{n=1}^{\infty} \{S(n) - 2S(n-1) + S(n-2)\} \frac{1}{n} \\ &= \sum_{n=1}^{\infty} S(n)\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\ &= \sum_{n=1}^{\infty} \frac{2S(n)}{n(n+1)(n+2)}.\end{aligned}$$

We divide the series into two parts

$$S_1 = \sum_{n=1}^{A-1}, \qquad S_2 = \sum_{n=A}^{\infty}.$$

Since

$$|S(n)| \leqq \sum_{a=1}^{n} \sum_{m=1}^{a} 1 = \frac{n(n+1)}{2},$$

it follows that

$$|S_1| \leqq \sum_{n=1}^{A-1} \frac{1}{n+2}.$$

If $A > d^{1/2}$ we have by Lemma 4

$$|S_2| < \sum_{n=A}^{\infty} \frac{nd^{1/2}}{n(n+1)(n+2)} = \frac{d^{1/2}}{A+1}.$$

Hence

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \left(\frac{d}{n}\right)\frac{1}{n}\right| &\leqq \sum_{n=1}^{A-1} \frac{1}{n+2} + \frac{d^{1/2}}{A+1} \\ &= \sum_{m=1}^{A-1} \frac{1}{m} - 1 - \frac{1}{2} + \frac{1}{A} + \frac{1}{A+1} + \frac{d^{1/2}}{A+1} \\ &\leqq \log (A-1) - \frac{1}{2} + \frac{1}{A} + \frac{d^{1/2}+1}{A+1}.\end{aligned}$$

Taking $A = [d^{1/2}]+1$ we have

$$\left|\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\right| \leqq \log d^{1/2} - \frac{1}{2} + \frac{1}{d^{1/2}} + \frac{d^{1/2}+1}{d^{1/2}+1}$$

$$= \frac{1}{2}\log d + \frac{1}{2} + \frac{1}{d^{1/2}} < \frac{1}{2}\log d + 1$$

since $d \geqq 5$.

THEOREM 1. *We have*

$$\log \epsilon < d^{1/2}((1/2)\log d + 1).$$

PROOF. It is known that the number $h(d)$ of classes of non-equivalent quadratic forms with determinant $d>0$, is given by

$$h(d) = \frac{d^{1/2}}{\log \epsilon}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\cdot$$

Since $h(d) \geqq 1$, we have the theorem.

THEOREM 2 (Schur). *We have*

$$\log \epsilon \leqq d^{1/2}\log d.$$

PROOF. For $d>e^2$, the theorem follows from Theorem 1. If $d<e^2$, then $d=5$. Evidently $\epsilon=(3+5^{1/2})/2$ and

$$\log \epsilon < 5^{1/2}\log 5.$$

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