ON PERRON INTEGRATION

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The definition of integral here presented had its origin in an unsuccessful attempt to establish the theorem on integration by parts[1] (Theorem 4.1 of this note) for the Perron integral, without detouring through the special Denjoy integral. In order to avoid the difficulties which I could not overcome, I amended the definition of the integral; the resulting definition I presented at the Oslo congress in 1936. Recently I have found a proof that the "new" integral is actually equivalent to that of Perron.

As compared with Perron's definition, the new definition has the slight disadvantage that it requires four associated functions instead of two; in all other respects the proofs of theorems for the Perron integral carry over unaltered. It has the advantage that it permits us to prove the general theorem on integration by parts (Theorem 4.1) without recourse to the deep-lying equivalence with the special Denjoy integral. This theorem is important, not only in itself, but because it also contains the corollary that if $f(x)$ is Perron integrable and $g(x)$ of limited total variation, their product is Perron integrable. Also it leads at once to the second theorem of the mean for Perron integrals.

1. Definition of the integral. Let $f(x)$ be a function defined on an interval $[a, b]$ and assuming values which are real numbers or $+\infty$ or $-\infty$. A set of four functions $\phi^i(x)$ $(i=1, 2, 3, 4)$ is a *tetrad adjoined to* $f(x)$ *on* $[a, b]$ if the following conditions are satisfied.

(1.1a) *The* $\phi^i(x)$ *are continuous on* $[a, b]$ *and all vanish at* $x=a$.

(1.1b) *For all except at most a denumerable collection of values of* x *the relations*

$$-\infty \neq D_+\phi^1, \qquad -\infty \neq D_-\phi^2,$$
$$+\infty \neq D^-\phi^3, \qquad +\infty \neq D^+\phi^4$$

are valid.

(1.1c) *For all except at most a denumerable collection of values of* x *the relations*

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[1] For a proof of the theorem with the added hypothesis that $g'(x)$ is finite except for a denumerable set, see R. L. Jeffery, *Perron integrals*, this Bulletin, vol. 48 (1942). pp. 714–717.

$$D_+\phi^1(x) \geqq f(x), \qquad D_-\phi^2(x) \geqq f(x),$$
$$D^-\phi^3(x) \leqq f(x), \qquad D^+\phi^4(x) \leqq f(x)$$

are valid.

In case the functions $\phi^i(x)$ also satisfy the conditions

$$|\phi^i(b) - \phi^j(b)| < \epsilon, \qquad i, j = 1, 2, 3, 4,$$

the tetrad will be said to be ϵ-adjoined to f. Functions satisfying the conditions imposed on $\phi^1, \phi^2, \phi^3, \phi^4$, respectively, will be called right majors, left majors, left minors, right minors, respectively.

(1.2) *The function $f(x)$ is P^*-integrable on $[a, b]$ if for every positive ϵ there is a tetrad ϵ-adjoined to f on $[a, b]$.*

Before defining the integral we establish an essential lemma.

LEMMA 1.1. *If ψ is a major function (right or left) for f, and ϕ is a minor function (right or left) for f, then $\psi-\phi$ is monotonic non-decreasing.*

Suppose first that ψ is a right major and ϕ a right minor. By (1.1b) and (1.1c), except on a denumerable set we have

$$D_+(\psi - \phi) \geqq D_+\psi - D^+\phi \geqq 0.$$

So[2] $\psi-\phi$ is non-decreasing. A similar proof applies if ψ is a left major and ϕ a left minor.

Suppose next that ψ is a right major and ϕ a left minor. By a theorem of G. C. Young,[3] the inequality $D_+\phi \leqq D^-\phi$ holds except at most on a denumerable set. So by (1.1b) and (1.1c) we have

$$D^+(\psi - \phi) \geqq D_+\psi - D_+\phi \geqq 0$$

except on a denumerable set, and[2] $\psi-\phi$ is non-decreasing.

A similar proof applies if ψ is a left major and ϕ a right minor.

COROLLARY 1.2. *If $f(x)$ is P^*-integrable on $[a, b]$, it is also P^*-integrable on $[a, x]$ for all x such that $a<x\leqq b$.*

For by Lemma 1.1 every tetrad which is ϵ-adjoined to f on $[a, b]$ is also ϵ-adjoined to f on $[a, x]$.

LEMMA 1.3. *If $f(x)$ is P^*-integrable on $[a, b]$, there is a function $F(x)$ on $[a, b]$ which is the* g.l.b. *of all right majors, the* g.l.b. *of all left majors, the* l.u.b. *of all left minors and the* l.u.b. *of all right minors. The function $F(x)$ is continuous.*

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[2] Saks, *Theory of the Integral*, p. 204.

[3] Hobson, *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 392.

Let ϕ be any minor of f. If ϕ^1 is a right major, for every x in $[a, b]$ we have

$$\phi^1(x) - \phi(x) \geqq \phi^1(a) - \phi(a) = 0;$$

so $\phi(x)$ is a lower bound for right majors. If we define $F(x)$ to be the g.l.b. of all right majors, then $F(x) \geqq \phi(x)$. Here ϕ is any minor, so F is an upper bound for all minors. If $\epsilon > 0$, there is a tetrad ϕ^i which is ϵ-adjoined to f, so

$$(1.4) \qquad F(x) - \phi^4(x) \leqq \phi^1(x) - \phi^4(x) \leqq \phi^1(b) - \phi^4(b) < \epsilon,$$

and likewise $F(x) - \phi^3(x) < \epsilon$. That is, $F(x)$ is the l.u.b. of all right minors and also is the l.u.b. of all left minors. Interchanging majors and minors shows that F is also the g.l.b. of all left majors.

If we let ϵ approach 0 through a sequence of positive values, by (1.4) the corresponding right minors $\phi^4(x)$ converge uniformly to $F(x)$, which therefore is continuous.

DEFINITION 1.4. *If $f(x)$ is P^*-integrable on $[a, b]$, and $F(x)$ is the function defined in Lemma* 1.2, *then the P^*-integral of f over $[a, b]$ is defined to be*

$$(P^*) \int_a^b f(x)dx = F(b).$$

Henceforth we omit the P^* before the integral.

COROLLARY 1.5. *With the hypotheses of Definition* 1.4, *if $a < x \leqq b$ then*

$$\int_a^x f(x)dx = F(x).$$

2. **Properties of the integral.** We can now readily establish the elementary properties of the P^*-integral; if f_1 and f_2 are P^*-integrable, so is kf_1 and so is $f_1 \pm f_2$ if defined (that is, if never of the form $\infty - \infty$), and the integrals have the values to be expected; f_1 is P^*-integrable over every subinterval of $[a, b]$, and is an additive function of intervals; and the function $f_0(x) = f_1(-x)$ is P^*-integrable over $[-b, -a]$, and its integral is equal to

$$\int_a^b f_1(x)dx.$$

The proofs differ from the corresponding proofs for the Perron integral only in minor details.

Furthermore, we can readily show that every P^*-integrable function is finite almost everywhere, is measurable, and is almost everywhere equal to the derivative of its indefinite P^*-integral. The proof in Sak's *Theory of the Integral*, p. 202, needs only obvious changes.

An alternative form of definition is given in the next theorem.

THEOREM 2.1. *Let $f(x)$ be defined on an interval $[a, b]$. In order that $f(x)$ be P^*-integrable on $[a, b]$ it is necessary and sufficient that for every positive number ϵ there exist four functions $\psi^i(x)$ $(i=1, 2, 3, 4)$ which satisfy* (1.1a) *and* (1.1b), *satisfy the inequalities in* (1.1c) *for almost all x, and are such that*

$$(2.1) \qquad | \psi^i(b) - \psi^j(b) | < \epsilon, \qquad i, j = 1, 2, 3, 4.$$

In this case, the P^*-integral of f over $[a, x]$ is the common g.l.b. *of all functions $\psi^1(x)$ and $\psi^2(x)$ and the common* l.u.b. *of all functions $\psi^3(x)$ and $\psi^4(x)$ which satisfy the conditions stated.*

The proof is essentially that given by Bauer[4] for the Perron integral. As an obvious corollary, if $f(x)$ is P^*-integrable on $[a, b]$ so is every function equivalent to f, and the integrals are the same.

3. **Substitution.** Next we establish the following theorem on change of independent variable.

THEOREM 3.1. *Let $f(x)$ be P^*-integrable on an interval $[a, b]$. Let $g(y)$ be a function defined on an interval $[\alpha, \beta]$ and possessing the following properties.*

(a) *$g(y)$ is continuous and non-decreasing on $[\alpha, \beta]$.*

(b) *$g(\alpha)=a$, $g(\beta)=b$.*

(c) *$\overline{D}g(y)<\infty$ except at most on a denumerable subset of $[\alpha, \beta]$. Then $f(g(y))\,\dot{g}(y)$ is P^*-integrable over $[\alpha, \beta]$, and*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(y))\dot{g}(y)dy.$$

For each positive ϵ there is a tetrad $[\phi^i]$ which is ϵ-adjoined to $f(x)$ on $[a, b]$. Define

$$\psi^i(y) = \phi^i(g(y)), \qquad i = 1, \cdots, 4; \alpha \leqq y \leqq \beta.$$

It is obvious that the ψ^i are continuous on $[a, b]$ and vanish at $y=\alpha$. Let E_1 consist of all y in $[\alpha, \beta]$ such that there exists a point y' with $y<y'\leqq\beta$ and $g(y')=g(y)$. The rest of the set $[\alpha, \beta]$ we

[4] H. Bauer, *Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen*, Monatshefte für Mathematik und Physik, vol. 26 (1915), pp. 153–198; also Saks, op. cit., p. 203.

denote by E_2; if y is in E_2 then $g(y')>g(y)$ for all y' such that $y<y'\leqq\beta$.

If y is in E_1, the function $g(y)$ remains constant on an interval $[y, y']$, so each $\psi^i(y)$ also is constant on $[y, y']$. Therefore

$$(3.1)\qquad -\infty < D_+\psi^1(y) = 0 = f(g(y))\dot{g}(y),$$

$$(3.2)\qquad +\infty > D^+\psi^4(y) = 0 = f(g(y))\dot{g}(y).$$

If y is in E_2, for each y' such that $y<y'\leqq\beta$ we have

$$(3.3)\qquad \frac{\psi^1(y') - \psi^1(y)}{y' - y} = \frac{\phi^1(g(y')) - \phi^1(g(y))}{g(y') - g(y)}\cdot\frac{g(y') - g(y)}{y' - y}\,.$$

By definition (1.1), $D_+\phi^1>-\infty$ except at most on a denumerable set X_0. If x is in X_0, the equation $g(y)=x$ can have at most one solution in E_2, for of two solutions lying in $[\alpha, \beta]$ the lesser is in E_1 by definition. So the set of such solutions is denumerable. To it we add the at most denumerable set on which $\overline{D}g=\infty$; the sum Y_0 is at most denumerable. If y is in E_2-Y_0, we let $y'\rightarrow y^+$, and from (3.3) obtain

$$(3.4)\qquad D_+\psi^1 \neq -\infty.$$

Moreover, if $g'(y)$ exists, as it does almost everywhere in $[\alpha, \beta]$, the same limiting process yields

$$(3.5)\qquad D_+\psi^1(y) = D_+\phi^1(g(y))\dot{g}(y).$$

Combining this with (3.1) and (1.1), we see that $D_+\psi^1(y)>-\infty$ except at most on a denumerable set and

$$D_+\psi^1(y) \geqq f(g(y))\dot{g}(y)$$

on almost all of $[\alpha, \beta]$.

The other functions ψ^i can be discussed analogously; we thus see that they satisfy the first three conditions of Theorem 2.1. The last condition (2.1) is trivial, since

$$\psi^i(\beta) - \psi^j(\beta) = \left| \phi^i(b) - \phi^j(b) \right| < \epsilon.$$

So by Theorem 2.1 the product $f(g(y))\dot{g}(y)$ is P^*-integrable on $[\alpha, \beta]$. Moreover, its integral over $[\alpha, \beta]$ lies between $\psi^4(\beta)$ and $\psi^1(\beta)$, that is, between $\phi^4(b)$ and $\phi^1(b)$. So does the integral of $f(x)$ over $[a, b]$. Since $\phi^4(b)$ and $\phi^1(b)$ differ by less than an arbitrary ϵ, the two integrals are equal.

Theorem 3.1 has an obvious analogue for monotonic non-increasing functions $g(y)$. Also, it is worth observing that the hypotheses on $g(y)$ force it to be absolutely continuous. For if we set $f\equiv 1$ in Theorem 3.1 we find that

$$g(\beta) - g(\alpha) = a - b = \int_a^b \dot{g}(y)dy.$$

Since g is non-negative, the integral on the right reduces to a Lebesgue integral, as we can show by the usual proof for the Perron integral. This equation implies the absolute continuity of $g(y)$.

4. **Integration by parts.** The principal reason for devising Definition 1.1 was to enable us to prove the following theorem.

THEOREM 4.1. *Let $f(x)$ be P^*-integrable on $[a, b]$, and let $F(x)$ be its indefinite P^*-integral. If $g(x)$ is of limited total variation on $[a, b]$, then the product $f(x)g(x)$ is P^*-integrable, and*

$$\int_a^b f(x)g(x)dx = F(b)g(b) - \int_a^b F(x)dg(x), \tag{4.1}$$

the last integral being a Stieltjes integral.

The function $g(x)$ can be written in either of the forms

$$g(x) = g_1(x) - g_2(x),$$
$$g(x) = g_3(x) - g_4(x),$$

where the g_i are positive, g_1 and g_2 are monotonic non-increasing and g_3 and g_4 are monotonic non-decreasing. Let ϕ^i be a tetrad ϵ-adjoined to f on $[a, b]$. We define sixteen functions by the formulas

$$\psi_j^i(x) = \phi^i(x)g_j(x) - \int_a^x \phi^i(x)dg_j(x), \qquad i, j = 1, 2, 3, 4. \tag{4.2}$$

It is evident that these all vanish at $x=a$. If x and x' are in $[a, b]$, we can apply the first mean-value theorem to the integral in (4.2) to obtain

$$\begin{aligned} \psi_j^i(x') - \psi_j^i(x) &= \phi^i(x')g_j(x') - \phi^i(x)g_j(x) - \phi^i(\bar{x})[g_j(x') - g_j(x)] \\ &= g_j(x')[\phi^i(x') - \phi^i(x)] \\ &\quad - [g_j(x') - g_j(x)](\phi^i(\bar{x}) - \phi^i(x)), \end{aligned} \tag{4.3}$$

where $\bar{x}$ is between x and x' (inclusive). Since ϕ^i is continuous and g_j is bounded, this shows that $\psi_j^i(x)$ is continuous.

Suppose that $g'(x)$ is defined and finite; this is true for almost all x. We divide both numbers of (4.3) by $x'-x$, writing the $x'-x$

under the square-bracketed terms on the right. The second term on the right tends to zero, so

$$D^+\psi_j^i(x) = g_j(x)D^+\phi^i(x), \tag{4.4}$$

and likewise for the three other Dini derivates.

If $\bar{x}\neq x$ equation (4.3) yields

$$\begin{aligned}\frac{\psi_j^i(x') - \psi_j^i(x)}{x' - x} &= g_j(x')\frac{\phi^i(x') - \phi^i(x)}{x' - x} \\ &\quad + (g_j(x) - g_j(x'))\cdot\frac{\phi^i(\bar{x}) - \phi^i(x)}{\bar{x} - x}\cdot\frac{\bar{x} - x}{x' - x}.\end{aligned} \tag{4.5}$$

This is still valid if $\bar{x}=x$, provided that we assign a value, say 0, to the middle factor in the last term. Henceforth we assume that x is in the set on which the inequalities in (1.1b) hold; this rejects at most a denumerable set of x. Let x' approach x from the right. The last factor in (4.5) is non-negative and bounded; so are the factors involving g_i if $j=1$ or 2. If $i=1$ the fractions involving ϕ^i are bounded below, since $D_+\phi^i(x) > -\infty$; if $i=4$ they are bounded above. Hence the left member is bounded below if $i=1$, above if $i=4$, and

$$D_+\psi_j^1 > -\infty, \quad D^+\psi_j^4 < \infty, \qquad j = 1, 2. \tag{4.6}$$

In a similar way, if $x'<x$ and $j=3$ or 4 we can establish

$$D_-\psi_j^2 > -\infty, \quad D^-\psi_j^3 < \infty, \qquad j = 3, 4. \tag{4.7}$$

Now we define

$$\begin{aligned}\psi^1 &= \psi_1^1 - \psi_2^4,\\ \psi^2 &= \psi_3^2 - \psi_4^3,\\ \psi^3 &= \psi_3^3 - \psi_4^2,\\ \psi^4 &= \psi_1^4 - \psi_2^1.\end{aligned} \tag{4.8}$$

From (4.6) and (4.7) we find that these satisfy (1.1b). From (4.4), together with (1.1b), we find that the inequalities of (1.1c) hold for almost all x, and (1.1a) has already been established.

Since F is the g.l.b. of majors and the l.u.b. of minors, we find

$$\psi_j^4(b) \leqq F(b)g_j(b) - \int_a^b F(x)dg_j(x) \leqq \psi_j^1(b), \qquad j = 1, 2.$$

Hence by (4.8)

$$\psi^1(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^4(b). \tag{4.9}$$

Likewise

$$\psi^2(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^3(b). \tag{4.10}$$

But

$$\begin{aligned} | \psi_j^1(b) - \psi_j^4(b) | &= | [\phi^1(b) - \phi^4(b)]g_j(b) - \int_a^b [\phi^1(x) - \phi^4(x)]dg_j(x)| \\ &\leqq \epsilon[| g_j(b) | + | g_j(b) - g_j(a) |], \end{aligned}$$

and, denoting the coefficient of ϵ by K_j, this implies

$$| \psi^1(b) - \psi^4(b) | < \epsilon(K_1 + K_2).$$

Likewise

$$| \psi^2(b) - \psi^3(b) | < \epsilon(K_3 + K_4).$$

Since ϵ is arbitrary, this implies that with (4.9) and (4.10) all four numbers $\psi^i(b)$ lie arbitrarily close to

$$F(b)g(b) - \int_a^b F(x)dg(x).$$

This establishes the inequality (2.1) of Theorem 2.1, hence proves the P^*-integrability of $f(x)$ $g(x)$, and also establishes equation (4.1).

5. **Equivalence of the P^*-integral and the Perron integral.** It is evident that every function which is[5] Perron integrable on an interval $[a, b]$ is also P^*-integrable, and the integrals are equal. For the Perron major functions serve simultaneously as right and left majors, and the Perron minors serve as right and left minors.

The converse is less evident. It requires slight generalizations of two known theorems.

LEMMA 5.1. *If $F(x)$ is a function which at all but at most a denumerable subset of a set M satisfies one of the inequalities*

$$D^+F < \infty, \qquad D^-F < \infty,$$
$$D_+F > -\infty, \qquad D_-F > -\infty,$$

then $F(x)$ is VBG_* on M.

The proof is essentially that given on p. 235 of Saks' *Theory of the Integral.* His equation (10.3) is replaced by

[5] By the Perron integral we mean the integral originally defined by Perron and studied by Bauer (loc. cit.); this is called the $\mathfrak{P}_0$ integral by Saks, and is equivalent to the $\mathfrak{P}$ integral (Saks, op. cit., p. 252).

$$0 < t - x < 1/n \text{ implies } [F(t) - F(x)]/(t - x) \leqq n;$$

the remainder of the proof is unaltered. The next lemma is a slight generalization of a theorem of Marcinkiewicz.

LEMMA 5.2. *If $f(x)$ is measurable on $[a, b]$, and has either a left major or a right major, and also has either a left minor or a right minor, then $f(x)$ is Perron integrable on $[a, b]$.*

The proof is that given by Saks, op. cit., p. 253; the principal change is that the reference to his Theorem 10.1 is replaced by a reference to our Lemma 5.1.

Since every P^*-integrable function $f(x)$ is measurable and has right majors and right minors, it is also Perron integrable by Lemma 5.2, and the equivalence of the integrals is established.

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ON THE LEAST PRIMITIVE ROOT OF A PRIME

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It was proved by Vinogradow[1] that the least positive primitive root $g(p)$ of a prime p is $O(2^m p^{1/2} \log p)$ where m denotes the number of different prime factors of $p-1$. In 1930 he[2] improved the previous result to

$$g(p) = O(2^m p^{1/2} \log \log p),$$

or more precisely,

$$g(p) \leqq 2^m \frac{p - 1}{\phi(p - 1)} p^{1/2}.$$

It is the purpose of this note, by introducing the notion of the average of character sums,[3] to prove that if $h(p)$ denotes the primitive root with the least absolute value, mod p, then

$$| h(p) | < 2^m p^{1/2};$$

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[1] See, Landau, *Vorlesungen über Zahlentheorie*, vol. 2, part 7, chap. 14. The original papers of Vinogradow are not available in China.

[2] Comptes Rendus de l'Académie des Sciences de l'URSS, 1930, pp. 7–11.

[3] The present note may be regarded as an introduction of a method which has numerous applications.