PERRON INTEGRALS

R. L. JEFFERY

At the recent conference on the theory of integration held at the University of Chicago Professor E. J. McShane remarked that formulas for integration by parts and for change of variable in the integrand have been established for the special Denjoy integral, and consequently for the Perron integral, since the two integrals are equivalent. He then raised the question as to the possibility of proving these formulas directly from the Perron definitions. Later, as we were waiting at the station for my train to leave, Professor Graves talked over with me the implications in McShane's point of view. The following solutions are the outcome of this conversation.

Integration by parts. *Let $g(x)$ be a continuous function on the interval (a, b) for which g' is finite except possibly for a denumerable set, and summable. Let the function $f(x)$ be integrable in the Perron sense and set*

$$F(x) = \int_a^x f(x)dx.$$

Then $f(x)g(x)$ is integrable in the Perron sense, and

$$\int_a^b f(x)g(x)dx = \left[F(x)g(x)\right]_a^b - \int_a^b F(x)g'(x)dx.$$

The conditions on g' make g absolutely continuous, and consequently the difference of two non-decreasing functions each of which is positive or zero. Hence we need consider only the case in which $g(x)$ is non-decreasing and positive or zero. Let $\psi(x)$ be a major function to $f(x)$. Consider $D_*(\psi g)$ which is the lower limit as $h \to 0$ of

$$\frac{\psi(x+h)g(x+h) - \psi(x)g(x)}{h}$$
$$= \frac{g(x+h)\{\psi(x+h) - \psi(x)\}}{h} + \frac{\psi(x)\{g(x+h) - g(x)\}}{h}.$$

From the continuity of g and the fact that both g and g' are not less than zero it follows that

$$D_*(\psi g) \geqq gD_*\psi + \psi g'$$

at every point for which g' exists. Then, since $D_*\psi > -\infty$ except for a

Received by the editors September 16, 1941, and December 11, 1941.

denumerable set, and $g' \geqq 0$ and finite except for a denumerable set, we have $D_*(\psi g) > -\infty$ except for a denumerable set. Furthermore, from $g' \geqq 0$, $\psi \geqq F$, and $D_*\psi \geqq f$ almost everywhere we get

$$D_*(\psi g) \geqq fg + Fg'$$

almost everywhere. These considerations together with the continuity of ψ and g show that ψg is a major function to $fg + Fg'$. Furthermore, if ϕ is a minor function to f it can be shown in a similar way that ϕg is a minor function to $fg + Fg'$. Again, if the difference $\psi - \phi$ is sufficiently small the difference $\psi g - \phi g$ is arbitrarily small. Then, since the common bound of ψg and ϕg is Fg, we conclude that $fg + Fg'$ is Perron integrable to Fg. The function F is continuous, and g' is summable. Hence Fg' is summable and, a priori, Perron integrable. Then fg being the difference of two Perron integrable functions is itself Perron integrable, and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b Fg'\,dx, \tag{1}$$

which is the desired result.

In the most general formula so far obtained[1] the only restriction on $g(x)$ is that it be of bounded variation. Then fg is Perron integrable and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b F\,dg. \tag{2}$$

In this case the foregoing procedure cannot be used for two reasons: The function ψg is not continuous at the discontinuities of g; if $g = +\infty$ at the points of a more than denumerable set and ψ is negative at the points of this set then $D_*(\psi g) = -\infty$ at the points of this more than denumerable set. Dropping the requirement that g be continuous, we prove the following result.

If $g(x)$ is of bounded variation and g' is finite except for a denumerable set, then formula (2) *holds.*

At the points of discontinuity of g it is necessarily true that g' is infinite, but this set is not more than denumerable. Our restriction is therefore that g' cannot be infinite at a more than denumerable set other than discontinuities. As in the proof of (1) there is no loss of generality in taking $g \geqq 0$ and non-decreasing. Let ψ be a major function to F with $\psi - F \leqq \epsilon$, and consider

[1] Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 201.

$$R(x) = \psi(x)g(x) - F(a)g(a) - \int_a^x \psi dg.$$

Using the law of the mean for Stieltjes integrals we have

$$\begin{aligned} R(x+h) - R(x) &= \psi(x+h)\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \\ &\quad + \psi(\xi)\{g(x+h) - g(x)\} \\ &= \{\psi(x+h) - \psi(\xi)\}\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \end{aligned}$$

where ξ is on the interval defined by the points x and $x+h$. The continuity of R then follows from that of ψ. Again

$$\begin{aligned} \frac{R(x+h) - R(x)}{h} &= \{\psi(x+h) - \psi(\xi)\}\frac{g(x+h) - g(x)}{h} \\ &\quad + g(x)\frac{\psi(x+h) - \psi(x)}{h}. \end{aligned}$$

The lower limit of the second term on the right is not greater than $g(x)D_*\psi > -\infty$ except for a denumerable set. The lower limit of the first term on the right can be $-\infty$ at points for which $g' = +\infty$ (h and ξ taking values which make $\psi(x+h) - \psi(\xi)$ negative). But $g' = +\infty$ for at most a denumerable set. Hence we have

$$D_*R > -\infty$$

except for a denumerable set. At a point for which g' is finite the limit of the first term on the right is zero. Hence we have

$$D_*R = gD_*\psi \geqq fg$$

almost everywhere. From these considerations it follows that R is a major function to fg. Also from $\psi - F < \epsilon$ we get

$$\theta(x) = \int_a^x \psi dg - \int_a^x F dg = \int_a^x \eta dg$$

where $0 \leqq \eta < \epsilon$. Hence $0 \leqq \theta(x) < \epsilon\{g(b) - g(a)\}$. From this and $\psi \geqq F$ we see that by taking a sequence of values of ϵ tending to zero we get a set of major functions $R(x)$ whose lower bound is

$$F(x)g(x) - F(a)g(a) - \int_a^x F dg.$$

Working in a similar way with functions ϕ which are minor to f we arrive at a set of functions which are minor to fg, and whose upper bound is the same as the lower bound of the set R. It then follows that fg is Perron integrable, and

$$\int_a^b fg dx = [Fg]_a^b - \int_a^b F dg. \tag{2}$$

Formula (2) includes formula (1), but a separate proof of (1) is of interest because of its simplicity.

Change of variable. *Let $f(x)$ be Perron integrable on (a, b), and let $x(t)$ be a continuous non-decreasing function of t on the interval (α, β) where $a = x(\alpha)$, $b = x(\beta)$, and on this interval let $x'(t)$ be finite except perhaps for a denumerable set. Then the function $f\{x(t)\}x'(t)$ is Perron integrable on this interval (α, β), and*

$$\int_a^b f(x) dx = \int_\alpha^\beta f\{x(t)\} x'(t) dt.$$

Let $\psi(x)$ be a major function to $f(x)$, and let $\chi(t) = \psi\{x(t)\}$. Then $\psi(t)$ is continuous. Also, since $D_*\psi > -\infty$ except for a denumerable set, and $x'(t) \geqq 0$ and finite except for a denumerable set, we have

$$D_*\chi(t) = D_{*x}\psi x'(t) > -\infty$$

except for a denumerable set. Again, since $D_{*x}\psi(x) \geqq f(x)$ almost everywhere, and $x'(t) \geqq 0$, we have

$$D_*\chi(t) \geqq f(x)x'(t) = f\{x(t)\}x'(t)$$

almost everywhere. From these results it follows that $\chi(t)$ is a major function to $f\{x(t)\}x'(t)$. If $\phi(x)$ is a minor function to $f(x)$ it can be shown in a similar way that $\Theta(t) = \phi\{x(t)\}$ is a minor function to $f\{x(t)\}x'(t)$. Furthermore, since the common bound of the functions $\psi(x)$ and $\phi(x)$ is $F(x)$, the common bound of the functions $\chi(t)$ and $\Theta(t)$ is $F\{x(t)\}$. Thus the function $f\{x(t)\}x'(t)$ is Perron integrable to $F\{x(t)\}$ and

$$F(b) - F(a) = \int_a^b f(x) dx = \int_\alpha^\beta \{x(t)\} x'(t) dt.$$

ACADIA UNIVERSITY