ON THE LOGARITHMIC MEANS OF REARRANGED PARTIAL SUMS OF A FOURIER SERIES

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Let $f(\theta)$ be a real, even and Lebesgue integrable function; let

$$f(\theta) \sim (1/2)a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta.$$

We write

$$s_0 = (1/2)a_0, \qquad s_n = (1/2)a_0 + a_1 + \cdots + a_n, \qquad n \geqq 1,$$

and denote by $s_0^*, s_1^*, \cdots, s_n^*$ the values of $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order. In 1935 Hardy and Littlewood [2][1] proved the following remarkable theorem:

THEOREM 1. *If*

$$f(\theta) = o\left(\log \frac{1}{\theta}\right)^{-1} \tag{1}$$

for small positive θ, *then*

$$\sum_0^n \frac{s_v^*}{v+1} = o(\log n). \tag{2}$$

Hardy and Littlewood gave two applications of this theorem by proving:

THEOREM 2. *If* (1) *holds, then*

$$\sum_1^n | s_v |^q = o(n) \tag{3}$$

for every positive q.

THEOREM 3. *If* (1) *holds and if*

$$a_n > -An^{-\xi} \tag{4}$$

for a positive A *and* ξ, *then* $s_n \to 0$.

They have also proved [1, Theorem 9] that in Theorem 3 the

Presented to the Society, December 30, 1941 under the title *On a theorem of Hardy and Littlewood*; received by the editors November 26, 1941.

[1] Numbers in brackets refer to the bibliography at the end of this paper.

assumption (1) can be replaced by

$$(5)\qquad F(\theta) \equiv \int_0^\theta |f(t)|\, dt = o\left(\frac{\theta}{\log (1/\theta)}\right),$$

but the proof requires a very difficult Tauberian theorem. Thus the question arose whether Theorem 1 remains true when (1) is replaced by (5).

In partial answer to this question we shall prove the following result.

THEOREM I. *If* (5) *holds and if for small positive* θ

$$(6)\qquad |f(\theta)| < \theta^{-c}, \qquad c \text{ a positive constant},$$

then (2) *holds.*

Note that condition (6) permits $f(\theta)$ to become as large as any power of $1/\theta$ near $\theta=0$.

For the proof of Theorem I we follow the device of Hardy and Littlewood. We suppose that $|s_m| = s^*_{v_m}$, $0 \leqq m \leqq n$, so that $v_0, v_1, \cdots, v_n$ is a permutation of $0, 1, \cdots, n$. We then have

$$S_n \equiv \sum_0^n \frac{s^*_m}{m+1} = \frac{2}{\pi}\int_0^\pi \frac{f(\theta)}{2 \sin \theta/2}\, g(\theta) d\theta,$$

where

$$g(\theta) = \sum_0^n \frac{\operatorname{sgn} s_m}{v_m + 1} \sin (m + 1/2)\theta.$$

We choose δ so that (6) holds for $0<\theta<\delta$, and write

$$S_n = \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi\right)\frac{f(\theta)}{\sin \theta/2}\, g(\theta) d\theta$$

$$\equiv I_1 + I_2 + I_3, \qquad \delta < 1 < n.$$

We first estimate I_1:

$$(7)\qquad \begin{aligned} |I_1| &\leqq \int_0^{1/n} \frac{|f(\theta)|}{\theta} |g(\theta)|\, d\theta \leqq \int_0^{1/n} |f(\theta)| \left(\sum_0^n \frac{m+1/2}{v_m+1}\right) d\theta \\ &\leqq (n+1/2)\left(\sum_{v=0}^n \frac{1}{v+1}\right)\int_0^{1/n} |f(\theta)|\, d\theta; \end{aligned}$$

hence, by (5)

$$I_1 = o(1) \qquad \text{as } n \to \infty.$$

Next

$$|I_3| = \left| \sum_0^n \frac{\operatorname{sgn} s_m}{v_m + 1} j_m \right|,$$

where

$$j_m = \frac{1}{\pi} \int_\delta^\pi \frac{f(\theta)}{\sin \theta/2} \sin (m + 1/2)\theta d\theta.$$

To any given $\epsilon > 0$ we can choose, by the Riemann-Lebesgue theorem, $m_0(\epsilon)$ so that $|j_m| < \epsilon$ for $m \geqq m_0$. Then

$$(8) \quad \begin{aligned} |I_3| &< \frac{1}{\pi} \frac{1}{\sin \delta/2} \int_\delta^\pi |f(\theta)| d\theta \cdot \sum_0^{m_0} \frac{1}{v_m + 1} + \epsilon \sum_{m_0}^n \frac{1}{v_m + 1} \\ &< \frac{m_0 + 1}{\pi \sin \delta/2} \int_\delta^\pi |f(\theta)| d\theta + \epsilon \log (n + 1), \end{aligned}$$

hence $I_3 = o$ $(\log n)$, as $n \to \infty$.

Theorem I is now reduced to showing that $I_2 = o$ $(\log n)$. We have from Young's inequality [**2**, p. 319]

$$uv \leqq lu \log^+ u + le^{(v-l)/l}$$

for $u > 0$, $l > 0$, and all real v. Here $\log^+ u = \max (0, \log u)$. We take $u = |f(\theta)/\sin \theta/2|$, $v = |g(\theta)|$, $l = 2$; then

$$\begin{aligned} |I_2| &< \frac{2}{\pi} \int_{1/n}^\delta \left| \frac{f(\theta)}{\sin \theta/2} \right| \log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| d\theta + \frac{1}{\pi} \int_{1/n}^\delta e^{|g(\theta)|/2} d\theta \\ &= K + L, \qquad \text{say}. \end{aligned}$$

Now from (6)

$$\log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| < \log \pi\theta^{-c-1} < (c + 1) \log \frac{\pi}{\theta},$$

hence

$$K \leqq 2(c + 1) \int_{1/n}^\delta |f(\theta)| \theta^{-1} \log \frac{\pi}{\theta} d\theta.$$

Integration by parts yields

$$(9) \quad K \leqq 2(c + 1) \left[F(\delta)\delta^{-1} \log \frac{\pi}{\delta} + \int_{1/n}^\delta F(\theta)\theta^{-2} \left\{ \log \frac{\pi}{\theta} + 1 \right\} d\theta \right],$$

and (5) gives $K = o$ $(\log n)$.

Finally, to estimate L, we use the following two lemmas (cf. Hardy and Littlewood [**2**]):

LEMMA A. *If*

$$\phi(\theta) = \sum_{-n}^{n} c_v e^{iv\theta}, \qquad \phi^+(\theta) = \sum_{-n}^{n} c_v^+ e^{iv\theta},$$

where c_v^+ *are the numbers* $|c_v|$ *rearranged so that* $c_0^+ \geqq c_{-1}^+ \geqq c_1^+ \geqq c_{-2}^+ \geqq \cdots$; *then*

$$\int_{-\pi}^{\pi} e^{b|\phi(\theta)|} d\theta \leqq 2 \int_{-\pi}^{\pi} e^{b|\phi^+(\theta)|} d\theta, \qquad \textit{for every } b > 0.$$

LEMMA B. $\left|\sum_1^n \cos v\theta/(v+1)\right| < \log (1/|\theta|) + c$ *for all* θ *and* n, *and a constant* $c > 0$.

Now, if we put

$$\frac{\operatorname{sgn} s_m}{v_m + 1} = \rho_m,$$

we have

$$g(\theta) = \Im(e^{i\theta/2}\gamma(\theta)), \qquad |g(\theta)| \leqq |\gamma(\theta)|,$$

where

$$\gamma(\theta) = \sum_{0}^{n} \rho_m e^{im\theta}$$

is a polynomial of degree n whose coefficients have (in some order) the absolute values $1, 1/2, \cdots, 1/(n+1)$. Hence, in the notation of the above lemma.

$$\gamma^+(\theta) = 1 + e^{-i\theta}/2 + e^{i\theta}/3 + e^{-2i\theta}/4 + \cdots,$$

the last term having modulus $1/(n+1)$ and an argument depending on the parity of n. The imaginary part of $\gamma^+(\theta)$ is bounded in n and θ, and the modulus of the real part is less than $\log (1/|\theta|) + c$, c a positive constant, due to Lemma B; hence $|\gamma^+(\theta)| < \log (1/|\theta|) + c$. We now obtain

$$(10) \qquad L < \frac{1}{\pi} \int_0^{\pi} e^{|\gamma(\theta)|/2} d\theta < \int_0^{\pi} e^{|\gamma^+(\theta)|/2} d\theta < \int_0^{\pi} \theta^{-1/2} d\theta + c;$$

the theorem now follows from (7), (8), (9), and (10).

THEOREM II. *To a given* $\delta > 0$ *denote by* $\nu(\delta, n)$ *the number of values* m *for which* $m \leqq n$, $|s_m| > \delta$; *then* $\log \nu = o(\log n)$ *as* $n \to \infty$.

We have

$$\delta \log \nu < \sum_0^n \frac{\overset{*}{s}_m}{m+1} = o(\log n), \qquad \text{from (2);}$$

δ being fixed our theorem follows [**2**, Theorem 3].

THEOREM III. *If* (5) *holds then* $s_n = o$ (log log n).

The proof is straightforward, using

$$s_n = \frac{1}{\pi}\int_0^\pi \frac{f(\theta)\sin(n+1/2)\theta}{\sin\theta/2}\,d\theta.$$

We have

$$|s_n| < \int_0^\pi \frac{|f(\theta)|\,|\sin(n+1/2)\theta|}{\theta}\,d\theta = \int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi$$

$$\equiv A_1 + A_2 + A_3,$$

where we choose $\delta = \delta(\epsilon)$ so that

$$F(\theta) < \frac{\epsilon\theta}{\log 1/\theta} \qquad \text{for } 0 < \theta < \delta.$$

Now $A_1 < (n+1/2)F(1/n) = o(1)$ as $n \to \infty$;

$$A_2 < \int_{1/n}^\delta |f(\theta)|\,\theta^{-1}d\theta = F(\theta)\cdot\theta^{-1}\Big]_{1/n}^\delta + \int_{1/n}^\delta F(\theta)\cdot\theta^{-2}d\theta$$

$$< \frac{\epsilon}{\log 1/\delta} + \epsilon\int_{1/n}^\delta \frac{d\theta}{\theta\log 1/\theta} = \frac{\epsilon}{\log 1/\delta} + \epsilon\log\log\frac{1}{\theta}\Big]_\delta^{1/n}$$

$$< \epsilon/\log 1/\delta + \epsilon\log\log n.$$

Hence for fixed ϵ and δ

$$\limsup_{n\to\infty}\frac{A_2}{\log\log n} \leqq \epsilon;$$

but ϵ is arbitrarily small, thus $A_2 = o$ (log log n). Finally

$$A_3 < \delta^{-1}\int_0^\pi |f(\theta)|\,d\theta = o(\log\log n);$$

this proves Theorem III.

THEOREM IV. *Denote by* $\phi(x)$ *a function satisfying the conditions* $\phi(0)=0$, $\phi(x)\uparrow$, *as* $x\uparrow$; $\phi(x) < \exp(e^{cx})$ *for all* $x>0$; *c a positive constant. Under the assumptions of Theorem* I, $\sum_1^n\phi(|s_m|) = o(n)$. *We may say the sequence* $\{s_n\}$ *is strongly summable relative to* $\phi(x)$.

We write for a given $\delta > 0$

$$\sum_1^n \phi(|s_m|) = \sum_{|s_m|<\delta} + \sum_{|s_m|\geqq\delta} \equiv B_1 + B_2;$$

then $B_1 < n\phi(\delta)$. By Theorems II and III we may write

$$|s_n| = \epsilon_n \log\log n, \quad \log \nu = \eta_n \log n, \qquad \text{where } \epsilon_n \to 0,\ \eta_n \to 0.$$

Now

$$B_2 < \nu\phi(\epsilon_n \log\log n) < n^{\eta_n}\cdot \exp\{(\log n)^{c\epsilon_n}\} = o(n),$$

hence

$$\limsup_{n\to\infty} \frac{1}{n}\sum_1^n \phi(|s_m|) \leqq \phi(\delta),$$

letting $\delta \to 0$ our theorem follows.

THEOREM V. *The assumptions* (4), (5), *and* (6) *imply* $s_n \to 0$.

This follows from Theorem II; see the proof of Theorem 5 in [**2**].

Final remark. In the more general case, when

$$f(\theta) \sim (1/2)a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta),$$

and when the point under consideration is θ_0, we use the familiar reduction. We have

$$(1/2)\{f(\theta_0+\theta) + f(\theta_0-\theta)\} \sim (1/2)\alpha_0 + \sum_1^\infty \alpha_n \cos n\theta,$$

where

$$\alpha_n = a_n \cos n\theta_0 + b_n \sin n\theta_0, \qquad n \geqq 1.$$

Suppose there is an s, for which

$$\int_0^\theta |f(\theta_0+t) + f(\theta_0-t) - 2s|\,dt = o\left(\frac{\theta}{\log 1/\theta}\right), \qquad \text{as } \theta \to 0,$$

and

$$|f(\theta_0+\theta) + f(\theta_0-\theta) - 2s| < 2|\theta^{-c}| \quad \text{for small } |\theta|.$$

Then application of Theorem I yields:

$$\sum_0^n \frac{1}{v+1} s_v^*(\theta_0) = o(\log n),$$

where $s_v^*(\theta_0), v=0, 1, \cdots, n$, is the sequence $|s_v(\theta_0)-s|$ in decreasing order.

References

1. G. H. Hardy and I. E. Littlewood, *Some new convergence criteria for Fourier series*, Annali, Scuola Normale Superiore, Pisa, (2), vol. 3 (1934), pp. 43–62.

2. ———, *Notes on the theory of series* (XVIII): *On the convergence of Fourier series*, Proceedings of the Cambridge Philosophical Society, vol. 31 (1935), pp. 317–323.

3. A. Zygmund, *Trigonometrical Series*, 1935.

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THE BASIC ANALOGUE OF KUMMER'S THEOREM[1]

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1. Introduction. About one hundred years ago, E. E. Kummer[2] proved the formula

$$(1) \qquad {}_2F_1\begin{bmatrix} a, & b; & -1 \\ 1+a-b & & \end{bmatrix} = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}$$

which has since been known as Kummer's theorem. This appears to be the simplest relation involving a hypergeometric function with argument (-1).

All the relations in the theory of hypergeometric series ${}_rF_s$ which have analogues in the theory of basic series[3] are those in which the argument is $(+1)$. Apparently, there has been no successful attempt to establish the basic analogue of any formula involving a function ${}_rF_s(-1)$. Since Kummer's theorem is fundamental in the proofs of numerous relations between hypergeometric functions of argument (-1), it seemed desirable that an attempt be made to prove the basic analogue of Kummer's theorem and to investigate the possibility of obtaining new relations in basic series with arguments corresponding to the argument (-1) in the classical case.

In this paper, the basic analogue of Kummer's theorem is obtained

Received by the editors November 27, 1941.

[1] The results presented in this paper are included in a dissertation for the doctorate, University of Nebraska, 1941.

[2] E. E. Kummer, *Ueber die hypergeometrische Reihe*, Journal für die reine und angewandte Mathematik, vol. 15 (1836), pp. 39–83.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, chap. 8.