ANALYSIS

255. C. R. Adams and A. P. Morse: *On approximating certain integrals by sums.*

For $f \in L(E)$, B a measurable subset of E, $0<|B|=\text{measure }(B)<\infty$, let $\mathfrak{M}_B f=\int_B f/|B|$. As B varies, let $\mathfrak{R}(f)$ represent the set of values of $\mathfrak{M}_B f$; and let ϕ be a function whose domain includes $\mathfrak{R}(f)$. For $0<\delta\leqq\infty$ let F be an arbitrary set-partition of E into disjoint measurable subsets each with diameter less than δ; and let the aggregate of all such partitions be denoted by $\Gamma_\delta(E)$. What conditions on f and ϕ will insure the (finite) existence of $\int_E\phi[f(x)]dx$ and of $\lim_{\delta\to 0}\inf_{F\in\Gamma_\delta(E)}\sum_{B\in F}\phi[\mathfrak{M}_B f]|B|$, $\lim_{\delta\to 0}\sup_{F\in\Gamma_\delta(E)}\sum_{B\in F}\phi[M_B f]|B|$ and their equality? For ϕ continuous, a necessary and sufficient condition is found. The hypothesis of continuity on ϕ cannot be dispensed with. "Sampling" can be allowed in the sum (see Adams and Morse, *Random sampling in the evaluation of a Lebesgue integral*, this Bulletin, vol. 45 (1939), pp. 442–447). A sufficient condition, often useful for testing, is found in terms of the existence of a convex dominant for $|\phi|$; such a convex dominant need not exist, but a condition is determined under which it does. Applications are made to functions f which are of bounded variation or are absolutely continuous in a certain generalized sense involving ϕ. Some new results in the general theory of functions of sets are included. (Received July 14, 1942.)

256. G. E. Albert: *Criteria for the closure of systems of orthogonal functions.*

Let the system F of functions $f_n(x)$, $n=0, 1, 2, \cdots$, be orthonormal on the interval (a, b). For any fixed point t in (a, b) let $g_t(x)$ denote the function which is equal to unity on (a, t) and zero on (t, b). Let $s_n(x)$ denote the partial sum of the generalized Fourier series with respect to F for the function $g_t(x)$. Define the function $\sigma_n(t)$ which, for each t in (a, b), is equal to $s_n(t)$. A necessary and sufficient condition that the system F be closed in the class of functions having integrable (Riemann or Lebesgue) squares on (a, b) is: $\lim_n\int_a^b|1-2\sigma_n(t)|dt=0$. A sufficient condition is that $\lim_n\int_a^b\{1-2\sigma_n(t)\}^2dt=0$. The verification of the latter criterion for the trigonometric system F is a matter of elementary calculus. Both criteria are extended to systems F orthogonal with respect to a positive weight function; in such cases the interval (a, b) may be infinite. The criteria stated follow easily from a theorem due to Vitali (Rendiconti dei Lincei, (5), vol. 30 (1921)). (Received June 6, 1942.)

257. R. H. Cameron and W. T. Martin: *Infinite linear difference equations with arbitrary real spans and first degree coefficients.*

The authors investigate the equation $\int_{-\infty}^{\infty}(z-\lambda)f(z-\lambda)dp(\lambda)+\int_{-\infty}^{\infty}f(z-\lambda)dq(\lambda)=g(z)$ in a strip $a<lmz<b$. Under fairly weak conditions on p, q, and g it is shown that the equation has a unique analytic solution of a fairly general character. (Received June 24, 1942.)

258. J. A. Clarkson and Paul Erdös: *On the approximation of continuous functions by polynomials.*

Let x^{n_i} be a set of powers of x, $n_i\to\infty$. Then a well known theorem of Müntz and Szász states that the necessary and sufficient condition that the powers x^{n_i} and 1 shall span the whole space of continuous functions, in the interval (0, 1) is that

$\sum_{i=1}^{\infty} 1/n_i = \infty$. It is proved that if $\sum 1/n_i < \infty$ and the continuous function $f(x)$ can be uniformly approximated in (0, 1) by polynomials in the x^{n_i}, then $f(x)$ can be continued to be analytic in the unit circle. The following result is also shown: Given $0 < a < b$; then the necessary and sufficient condition that the powers x^{n_i} shall span the space of all continuous functions in (a, b) is that $\sum 1/n_i < \infty$. (Received July 3, 1942.)

259. Mark Kac: *On the distribution of values of trigonometric sums with linearly independent frequencies.*

Let $f(t) = \sum_{k=1}^{n} a_k \cos 2\pi\lambda_k t$, where the a's are real and the λ's real and linearly independent. Denote by $N_T(a)$ the number of t's in $(-T, T)$ for which $f(t) = a$. It is proved that $E(a) = \lim N_T(a)/2T$ as $T \to \infty$ exists and that $E(a) = 2\pi^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cos a\xi K(\xi,\eta) d\eta d\xi/\eta^2$, where $K(\xi,\eta) = \prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k|(\xi^2 + 4\pi^2\lambda_k^2\eta^2)^{1/2})$. This, in a way, completes an earlier investigation of E. R. van Kampen, A. Wintner, and the present author. (See American Journal of Mathematics, vol. 61 (1939), pp. 985–991, in particular section 3.) (Received July 23, 1942.)

260. Ella Marth: *On Garvin's F-series.* Preliminary report.

A generalized Lambert series of the form $F(z) = \sum a_n z^{\lambda n}/(1 - z^{\mu n})$ was defined and studied by Garvin (M. C. Garvin, *A generalized Lambert series*, American Journal of Mathematics, vol. 58 (1936), pp. 507–513). With certain restrictions on the coefficients the series was found to have the unit circle as a natural boundary. In obtaining this natural boundary Garvin used a radial approach to points on the boundary. Furthermore she found that for a rational point z_0 on the boundary (1) $\lim_{z \to z_0} \{(1 - z/z_0) F(z)\}$ equals $(1/\mu) \sum a_{l\nu}/l\nu$ with certain reservations. In this paper the approach to the boundary is extended to complex approach for which (1) holds under specific conditions. For the case when z_0 is an irrational point (1) becomes zero for both radial and complex approach. (Received July 21, 1942.)

261. Josephine M. Mitchell: *On double Sturm-Liouville series.*

From the Sturm-Liouville system $\{\phi_m(x)\}$ $(m = 0, 1, 2, \cdots; 0 \leqq x \leqq \pi)$, the double Sturm-Liouville series of a function $f(x, y)$, integrable (L) over $Q(0 \leqq x \leqq \pi, 0 \leqq y \leqq \pi)$, namely: $\sum_{m,n=0}^{\infty} a_{mn}\phi_n(x)\phi_n(y)$, $a_{mn} = \int_0^{\pi}\int_0^{\pi} f(s, t)\phi_m(s)\phi_n(t) ds\, dt$, is formed and the equiconvergence and equisummability of this series with the Fourier cosine-cosine series of $f(x, y)$ is considered. Haar's theorem (1910) for the one variable case that the Sturm-Liouville series of an integrable function is uniformly equiconvergent on $(0, \pi)$ with its Fourier cosine series does not generalize; only a weaker theorem on equiconvergence is proved. The main result is, however, that the double Sturm-Liouville series of $f(x, y)$ is $(C, 1, 1)$ *equisummable with its Fourier cosine-cosine series at all points* (x, y) *in* Q *for which* $(1/hk)\int_0^h\int_0^k |f(x+s, y+t)| ds\, dt$ $(0 < |h| \leqq \pi, 0 < |k| \leqq \pi)$ *is bounded.* A summability theorem, similar to one for double Fourier series (B. Jessen, J. Marcinkiewicz and A. Zygmund, 1935) stating that *if* $f \log^+ |f|$ *is integrable, the double Sturm-Liouville series of* $f(x, y)$ *is* $(C, 1, 1)$ *summable to* $f(x, y)$ *almost everywhere* follows readily. Finally applying to the Sturm-Liouville system a modification of the Poisson method of summation, introduced by S. Bergmann (1941), it is proved that the double Sturm-Liouville series of any integrable function is equisummable by this method with its Fourier cosine-cosine series. (Received July 23, 1942.)

262. K. L. Nielsen and B. P. Ramsay: *On particular solutions of linear partial differential equations.*

A method for the solution of boundary value and characteristic value problems consisting of approximations by expressions $W_n = \sum_{\nu=1}^{n} \alpha_\nu^{(n)} \phi_\nu(x, y)$, where $\phi_\nu(x, y)$ are particular solutions of the considered differential equation, has been given by Bergman (Duke Mathematical Journal, vol. 6 (1940), pp. 537–561). In applying this method it is important for practical purposes to obtain a simple procedure for the construction of the particular solutions and in this connection Bergman (Mathematicheskii Sbornik, vol. 44 (1937), pp. 1169–1197) has proved that to every equation $L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0$, where a, b, c are functions of $z = x + iy$ and $\bar{z} = x - iy$ and the subscripts denote the partial derivatives, there exist functions $E(z, \bar{z}, t)$ such that $P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z(1-t^2)/2) [1-t^2]^{-1/2} dt$, where f is an arbitrary analytic function of one complex variable, will be a particular solution of $L(U) = 0$. To expedite the numerical computation for practical problems it is further desired that E has a simple structure so that the integral $P(f)$ may be easily evaluated for arbitrary values of z and $\bar{z}$. In this note the authors determine certain types of equations, $L(U) = 0$, for which E has the simple form $E(z, \bar{z}, t) = \exp(Nt^n + Mt^m)$, where $N = N(z, \bar{z})$ and $M = M(z, \bar{z})$. The authors further show that M and N may be determined from the coefficients a, b, c. (Received June 22, 1942.)

263. Raphael Salem: *On a theorem of Zygmund.*

The following theorem is due to Zygmund: If a continuous function $f(x)$ of bounded variation and of period 2π has a modulus of continuity $\omega(\delta)$ such that the series $\sum n^{-1} [\omega(n^{-1})]^{1/2}$ converges, then the Fourier series of $f(x)$ is absolutely convergent. The purpose of the present paper is to prove that the exponent 1/2 of this theorem is the best possible one: the theorem becomes false if the exponent 1/2 is replaced by $\epsilon + 1/2$, ϵ being any positive number. (Received August 1, 1942.)

264. Otto Szász: *On the partial sums of Fourier series at points of discontinuity.*

In the first part of this paper the summability method $T_n = \sum_1^n a_{n\nu} \tau_\nu$ with $a_{n\nu} = \rho_n^\nu \nu^{-1} \sin \nu\theta_n$, $\nu = 1, 2, \cdots, n$, $\rho_n \to 1$, $\theta_n \to 0$, is considered. Necessary and sufficient conditions are given for permanency of this transform relative to Cesàro summability of the sequence $\{\tau_n\}$, of some integral order. In the second part application to Fourier series yields a Gibbs' phenomenon for rather general classes of functions. To quote one such case: If $f(\theta) \sim \sum b_n \sin \theta$, $\sum_1^n \nu b_\nu > -pn$, for some $p > 0$, and for all $n > 0$, and $(2/\pi) \int_0^\theta |f(t) - j/2| dt \to 0$, for some j, then for any positive integer λ: $\sum_1^n b_\nu \sin \nu\theta_n \to \int_0^{\lambda\pi} (\sin t)/t \, dt$, whenever $n\theta_n - \lambda\pi = O(n^{-1})$ as $n \to \infty$. (Received July 15, 1942.)

265. C. J. Thorne: *An Appell subset.*

Some of the properties and applications of the polynomials defined as follows are given: $\int_a^b \phi_n^{[r]}(x) dx = \delta_n^r$; where $\delta_n^r = 0$ for $n \neq r$ and $= 1$ for $n = r$, n-degree of polynomial, $[r]$ — rth derivative. In particular this polynomial set is shown to be an Appell subset and to satisfy difference equations similar to those satisfied by the Bernoulli polynomials which lead to the Euler-MacLaurin expansion formula. Use of the Appell subset in the determination of functions defined in various ways is illustrated. (Received July 30, 1942.)

266. Hermann Weyl: *On Hodge's theory of harmonic integrals.*

Hodge's fundamental existence theorem for harmonic integrals on Riemannian manifolds of any dimensionality is proved by the parametrix method. (The proof incorporated in Hodge's recent book on *Harmonic Integrals*, Cambridge, 1941, is wrong.) (Received July 1, 1942.)

267. Hassler Whitney: *Differentiability of the remainder term in Taylor's formula.*

If $f(x)$ is of class C^m, and $1 \leqq n \leqq m$, then $f(x) = \sum_{i=0}^{n-1} f^{(i)}(0)x^i/i! + x^n f_n(x)/n!$. It is shown that $f_n(x)$ is of class C^{m-n}, but not necessarily of higher class, and $\lim_{x\to 0} x^k f^{(m-n+k)}(x) = 0$ $(k=1, \cdots, n)$. A converse is true. A similar theorem holds in more dimensions. (Received July 28, 1942.)

268. Hassler Whitney: *Note on differentiable even functions.*

It is shown that an even function $f(x)$ of class C^{2s} (or class C^∞, or analytic) may be written as $g(x^2)$, with g of class C^s (or class C^∞, or analytic). (Received July 28, 1942.)

269. Hassler Whitney: *The general type of singularity of a set of* $2n-1$ *smooth functions of n variables.*

Let f be a mapping of class C^1 of an n-manifold M^n into an M^{2n-1}. Then arbitrarily near f is a mapping f', regular except at isolated singular points; at each of these, a certain condition (C) holds. (C) involves first and second derivatives, but is independent of the coordinate system employed. If (C) holds at p, and the mapping is of class C^{4r+8} (or class C^∞, or analytic), then coordinate systems about p and $f(p)$, of class C^r (or class C^∞, or analytic), exist such that the mapping is exactly $y_1 = x_1^2$, $y_i = x_i$, $y_{n+i-1} = x_1 x_i$ $(i=2, \cdots, n)$. (Received July 28, 1942.)

Applied Mathematics

270. Stefan Bergman: *Operators in the theory of differential equations and their application.* I.

By introducing $u = x \cos\theta + y \sin\theta$, $v = -x \sin\theta + y \cos\theta$ and $\xi = (\sigma/2k) + \theta$, $\eta = (\sigma/2k) - \theta$, where $\sigma_x = \sigma + k \sin 2\theta$, $\sigma_y = \sigma - k \sin 2\theta$, $\tau_{xy} = -k \cos 2\theta$ the equations of the theory of plasticity can be written in the form $(\partial^2 u/\partial\xi\partial\eta) - u/4 = 0$, $(\partial^2 v/\partial\xi\partial\eta) - v/4 = 0$ (see Geiringer and Prager, Ergebnisse der exakten Naturwissenschaften, vol. 13, p. 350). Here $\sigma_x, \sigma_y, \tau_{xy}$ are stresses, x, y, cartesian coordinates. Particular solutions of these equations can be written in the form $u(\xi, \eta) = \int_{-1}^{1} \exp(t(\xi\eta)^{1/2}) \{f[\xi(1-t^2)/2] + g[\eta(1-t^2)/2]\}(1-t^2)^{1/2} dt$ where f and g are arbitrary twice continuously differentiable functions of one variable. (See Duke Mathematical Journal, vol. 6 (1940), pp. 538 and 557.) This class of functions possesses a base $\{u_\nu(\xi, \eta)\}$ such that each u_ν satisfies two (simple) ordinary linear differential equations of second order with rational coefficients. Entire solutions u_ν of the above partial differential equation are such that every u defined in a convex domain can be approximated by sums of the form $\sum_{\nu=1}^{n} a_\nu^{(n)} u_\nu$. The author indicates an approximation procedure of a function u given by its boundary values. These functions u possess singularities which can be characterized in a way analogous to that in Comptes Rendus de l'Académie des Sciences, vol. 205 (1937), pp. 1360–1362. (Received June 3, 1942.)