The reader—specialist in topology or not—will find particularly interesting the elementary proof of Brouwer's fixed-point theorem for closed n-cells and the use of this theorem in showing that the dimension of E_n (Euclidean n-space) is n; Hurewicz's beautiful proof that every n-dimensional space can be imbedded in E_{2n+1} and the use of the technical devices in that proof in establishing the relation between dimension and the orders of coverings; the systematic use of the notion of extension (of a mapping or homomorphism); a separation theorem for homeomorphs of $(n-1)$-spheres in E_n as a corollary of a theorem about mappings; a discussion of the relation between dimension and measure; a number of remarks about spaces of infinitely many dimensions; an exposition of Čech homology theory and the dual cohomology theory; the use of cohomologies in questions about mappings and their extensions. It is understood throughout the book that "space" means "separable metric space." A short appendix contains an account of the difficulties which arise in extending the theory to more general spaces.

P. A. SMITH

The Laplace Transform. By David Vernon Widder. (Princeton Mathematical Series, no. 6.) Princeton University Press, 1941. 10+406 pp. $6.00.

The theory of integral operators has developed into a major branch of analysis. It has proved to be a valuable, in fact essential, complement to the theory of differential equations, while its logical structure is more satisfactory in the sense that one deals in general with equivalences, rather than the sufficient conditions of differential equations.

As one might expect, the theory of integral operators, even in the linear case, also has complementary subdivisions. There is the well known spectral theory, which while highly effective when applicable, seems to be closely confined to functions in L_2. For other classes of functions, there have been investigations of particular operators, for example the Fourier transform.

But besides the Fourier transform, there are others which have been investigated for more than a century and it is to certain of these that the present work is devoted. However, the author also presents various applications of integral transforms (including the Fourier) of great interest and importance so that this work attains a generality which makes its subject matter of basic importance for the analyst.

The book is concerned with the following transforms: the Laplace transform

$$(*) \qquad f(s) = \int_0^\infty e^{-st} d\alpha(t),$$

the bilateral Laplace transform (*′) in which the lower limit of integration is not zero but minus infinity, the Mellin transform which is obtainable by a change of variable from the bilateral transform, the Stieltjes transform

$$(**) \qquad f(s) = \int_0^\infty \frac{d\alpha(t)}{s+t},$$

which is, formally, the iterate of (*), and the Fourier transform in certain applications. $\alpha(t)$ is, at least, of bounded variation on every finite interval not containing the origin but during the course of the investigation, it is specialized in various ways, for instance, to be of bounded variation, or to be the integral of a function ϕ of a specified class, L, L_2, L_p, or M.

The integration in (*) or (**) is Riemann-Stieltjes and Chapter I contains a rather complete discussion of this type of integration and of functions of bounded variation, including Helly's theorem. If s is regarded as a complex variable, the domains of existence and analyticity of $f(s)$ may be specified by certain convergence considerations given in the first part of Chapter II for (*), the first part of Chapter VI for (*′) and the first part of Chapter VIII for (**).

Besides the important applications, which we will discuss below, there are two interrelated problems treated for these transforms. One of these is the problem of "inversion," that is, given f, to find α. There are, in general, for any such transform at least two such inversion theories. The other is the "representation problem" which may be described as the problem of finding the class of f's, which corresponds to a given class of α's. In general, one can readily guess at the class of f's, but the difficulty arises in trying to show that each f in the class has the specified representation.

The "complex" inversion theory is concerned with formulas which express α in terms of $f(s)$ regarded as a function of a complex variable s. For (*) and (*′), as given in Chapters II and VI, the formula involves integration along a line parallel to the axis of imaginaries and the theory has similarities with that of the Fourier transform. For (**), we have in Chapter VIII, a theory somewhat like that of the Poisson integral. For (*), this inversion formula yields a representation theory for those α's which are integrals of functions in L_2. In

Chapter II, one also finds certain other representation discussion for (*) in the case in which $f(s)$ is regarded as a function of a complex variable.

The "real" inversion theory is given in Chapter VII for (*) and in Chapter VIII for (**). It involves the values of all the derivitives of $f(s)$ along the positive real axis. For (*) in the case in which α is absolutely continuous the result is due to E. L. Post and has been extended by Widder. Formulas for the variation of α and for the saltus at a point are given. The associated representation theory is more highly developed and considers the class of α's of bounded variation, the class of non-decreasing α's, the bounded subclass of these, the class of integrals of a $\phi \in L_p$ for a fixed $p \geqq 1$, and the class of integrals of a $\phi \in M$. The relation between (*) and (**) is used in Chapter VIII, §25, to obtain a real inversion formula for (*), involving just the values of f itself.

The most interesting representation problem for (*) is undoubtedly that in which α is increasing. Under these circumstances, $f(s)$ is "completely monotonic," that is for $s \geqq 0$, $f^{(k)}(s)(-1)^k \geqq 0$. The converse, that every completely monotonic f is the Laplace transform of a monotonic bounded $\alpha(t)$, is called Bernstein's theorem. This result was also obtained in a different form by Hausdorff and independently by Widder.

In preparation for this result, three moment problems are discussed and solved in Chapter III. A moment problem is one in which, given a sequence $\{\mu_n\}$ one is required to find a monotonic $\alpha(t)$, such that

$$\mu_n = \int_a^b t^n d\alpha(t).$$

If $a = 0$, $b = 1$, we have the Hausdorff moment problem; if $a = 0$, $b = \infty$, the Stieljes moment problem and if $a = -\infty$, $b = \infty$, the Hamburger moment problem. In the Hausdorff case, one may even consider the more general case in which α is of bounded variation instead of monotonic, and the result is unique. In the other cases if α is not monotonic the answer is not unique. The Hausdorff problem is associated with the process of summation of series and permits one to solve equivalence and comparison questions for the usual methods of Cesàro and Hölder. Those series $\{\mu_n\}$ for which α is monotonic are "positive definite" and this is related to the corresponding notion for quadratic forms. The determinant criterion for positive definiteness appears and is used in a discussion of certain work of Boas on those cases of the moment problem in which the limits of integration are infinite.

In Chapter IV, three proofs of Bernstein's theorem are given. Two of these are based on the solution of the Hausdorff moment problem given in the preceding chapter, the remaining one involves the Laguerre polynomials. Since the set of completely monotonic functions is multiplicative, the same is true for the set of functions f in the form (*). The chapter concludes by discussing "completely convex" functions for which $(-1)^k f^{(2k)}(s) \geqq 0$ for $s \geqq 0$ and these are shown to be entire.

Chapter V discusses various limit questions for integrals and certain striking applications. Suppose $f(s)$ is defined by (*) for $s>0$. Then if $\lim_{t\to\infty} \alpha(t)$ exists and equals A, $\lim_{s\to 0} f(s)$ exists and equals A. The converse is true only if α is suitably restricted and this converse is one of the Tauberian theorems treated in Chapter V and including results on (**). These results introduce two famous theorems. The first is Wiener's general Tauberian theorem on the equation

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} g(x-t)p(t)dt = A\int_{-\infty}^{\infty} g(t)dt$$

for a fixed bounded p. This theorem states that if this equation holds for a $g \in L$ whose Fourier transform does not vanish, it holds for all $g \in L$. A number of variants of this theorem are also shown. The second is the prime number theorem for which two proofs are given, one due to Wiener, the other due to Ikehara.

The notion of a resultant is treated for a number of transforms. For instance if we let

$$\gamma(x) = \int_{-\infty}^{\infty} \alpha(x-t)\beta(t)dt,$$

it is shown in Chapter VI, that for both the Fourier and bilateral Laplace transform, the product of the transforms of α and β is the transform of the result, γ. The book concludes with a discussion of the integral equation

$$f(s) = \lambda \int_0^{\infty} \frac{f(t)dt}{s+t}$$

and the corresponding equation for the Laplace case.

The book is certainly enjoyable and interesting. The style is clear, there are few typographical errors and the subject matter is increasingly impressive as one reads on. The applications are particularly striking. In some cases, it is not clear why so many proofs of the same theorem are given and a guide to a reader who might be interested in any one of the many specific results would be valuable. However, it would be quite easy to use various topics treated in the book in a course, whose main interest is not integral operators. One might mention, the Riemann-Stieltjes integral, functions of bounded variation, methods of summation of series, positive definite series, the moment problems, Bernstein's theorem, the Tauberian theorems, the prime number theorem, the Laguerre polynomials, the notion of a positive definite kernel of an integral equation, and the specific integral equations mentioned.

Thus the author has presented us with a treatise on a branch of analysis of great importance and whose applications are of wide interest. The book is extremely satisfactory, when concerned with either its principal topics or the other related developments and one is confident that it will have a most valuable effect both on research and graduate study.

F. J. MURRAY

Mathematical Methods in Engineering. By Theodore von Kármán and Maurice A. Biot. New York, McGraw-Hill, 1940. 12+505 pp. $4.00.

This book by two masters of applied mathematics selects certain representative groups of advanced engineering problems and presents the appropriate mathematical methods of solution, together with helpful excursions into purely mathematical topics. This is a very effective plan that might well be followed by future books in applied mathematics. The problems are largely in mechanics and are interesting, instructive and up-to-date, especially those on the airplane—as might be expected from the interests and accomplishments of the senior author.

Besides the problems worked out in the text, each chapter includes a set of problems of graded difficulty to be worked out by the student, the answers being given at the end of the book. Each chapter opens with a thought-provoking quotation from an authority in the field and ends with a well selected list of references, mostly standard texts.

The authors are as careful with their mathematics as with their physics and engineering, but have chosen to omit mathematical proofs in many cases where they felt that space was not available. It seems to the reviewer that in some of these cases at least a brief outline of the proof could have been given to the advantage of the reader.

Chapter I is a direct, clear introduction to ordinary differential