A NEW PROOF OF THE CYCLIC CONNECTIVITY THEOREM

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The cyclic connectivity theorem was first proved for the plane in 1927 by G. T. Whyburn [5]. The extension of this theorem to metric space afforded some difficulty and the first proof [1] was long and tedious and complicated with convergence difficulties. A second and simpler proof appeared in 1931 [6], but in this proof it is necessary that quite a number of properties of Peano spaces be proved in advance.

This note attempts to give a new proof in which convergence troubles are encountered at just one point (step (b)) and in which just three theorems about Peano space need be known in advance: (A) *Every component of an open set is open.* (B) *Open connected sets are arc-wise connected.* (C) *The space is arc-wise locally connected.* Actually just two properties need to be established before cyclic connectivity can be proved, for the third theorem (C) is a simple consequence of the first two.[1] Thus the cyclic connectivity theorem may be established at the very beginning of the theory of Peano spaces and is available for use in studying other properties.

CYCLIC CONNECTIVITY THEOREM. *If no single point of a locally compact, connected and locally connected metric space separates the space between the two given points, there is a simple closed curve containing the two points.*

Let p and q be the two points. There exists an arc α of the space S with end points p and q by (B). We shall say that an arc β *spans* the point v of α if β has only its end points on α and v lies between these end points. We shall say that a set of arcs C *spans* a subset K of α if each point of K is spanned by some arc of the set C.

If an arc β exists with end points r and q and such that $\alpha\cdot\beta=r+q$, then step (d) in the proof has been achieved. Hence we consider only the case where no such arc exists. This assumption is used in the proof of step (b).

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[1] Let G be the component of $S(p, \epsilon)$ containing the point p. By (A), G is open. Then for some δ, $S(p, \delta)\subset G$. By (B) G is arc-wise connected. Hence every point of $S(p, \delta)$ may be joined to p by an arc in G, and thus in $S(p, \epsilon)$, which proves arc-wise local connectivity.

(a) *If r is a point of $\alpha-p-q$, there is an arc β spanning r.*

As r does not cut S between p and q, by (A) p and q belong to the same component C of $S-r$. Then C contains an arc γ with end points p and q by (A) and (B). On γ in the order from p to q there is a first point y of the subarc rq of α. On γ in the order y to p there is a first point x of the subarc rp of α. Then the subarc xy of γ is the desired arc β.

For any point r, let $f(r)$ be the greatest lower bound of the diameters of β, for all arcs β spanning r.

(b) *$f(r)\to 0$ as $r\to q$.*

If this is false, there is an $\epsilon>0$ and a sequence of points of α converging to q so that no one of these points may be spanned by an arc of diameter less than ϵ. We may choose ϵ so small that $S(q, \epsilon)$ is compact. Let us select a spanning arc for each point of the sequence. Infinitely many of these arcs are distinct since we assumed in the beginning that no one of them has q as an end point. Hence there exists an infinite set of arcs $\beta_1, \beta_2, \beta_3, \cdots$ such that (i) $\alpha\cdot\beta_i=x_i+y_i$, the end points of β_i, and x_i precedes y_i on α, (ii) diam $\beta_i\geqq\epsilon$, (iii) β_i spans a point p_i of α which cannot be spanned by any arc of diameter less than ϵ, (iv) we have the order $pp_1y_1p_2y_2\cdots q$ on α, (v) $y_i\to q$ and $\rho(y_i, q)<\epsilon/6$. On β_i in the order y_i to x_i let z_i be the first point such that $\rho(z_i, q)=\epsilon/3$. The set of points $\{z_i\}$ has a limit point z and we may assume $z_i\to z$. By (C) there exist arcs z_iz of diameter less than $\epsilon/6$ for i sufficiently large. Then $y_iz_i+z_iz+z_{i+1}z+y_{i+1}z_{i+1}$ contains an arc of diameter less than ϵ spanning p_{i+1}, which is a contradiction.

(c) *If r is any point of $\alpha-p-q$, there exists a countable sequence of arcs $\{\beta_i\}$ spanning the set $rq-q$ and such that* diam $\beta_i\to 0$.

Let $p_i\to q$ with the order $prp_1p_2\cdots q$. Let $p_0=r$ and α_i be the subarc $p_{i-1}p_i$ of α. From (a) there exists a family of arcs spanning α_i. From (b) these may be chosen of arbitrarily small diameter for i large. Using the Borel theorem there is a finite subfamily spanning α_i. The set of all these finite subfamilies is a countable set with the desired properties.

(d) *There is a simple closed curve containing q.*

Let u_i and v_i be the end points of β_i and suppose the order pu_iv_iq on α. From the method in which the arcs β_i were chosen we see that only a finite number span any one point and no one intersects more than a finite number of others. Of the finite set spanning r, choose the one whose end point v_i is nearest q on α in the order p to q. We may assume this is β_1. Of the finite set, if nonvacuous, of arcs β_i which intersect $\beta_1-u_1-v_1$, choose the one whose end point v_i follows all others on α. We may assume this is β_2. Of the finite sets, if nonvacuous, which intersect $\beta_2 - u_2 - v_2$, we choose the one whose end point v_i follows all others on α, and we may suppose this is β_3. This process must terminate at some finite step for otherwise $\sum\beta_i$ would contain an arc having u_1 and q as end points and containing no other point of α.[2] This arc together with the subarc u_1q of α would give the desired simple closed curve.

If the process terminates at β_k, then $\sum_1^k\beta_i$ contains an arc γ_1 with end points $s_1 = u_1$ and $t_1 = v_k$ and containing no other point of α. We now define an arc γ_2 with end points s_2 and t_2 using exactly the process by which γ_1 was defined but starting with the arcs spanning v_k instead of r. We continue and define $\gamma_3, \gamma_4, \cdots$. The arcs γ_i are mutually exclusive except that t_i may coincide with s_{i+2}. On α we have the order $s_1 < s_2 < t_1 \leqq s_3 < t_2 \leqq s_4 < \cdots < q$. Then the desired two arcs forming a simple closed curve containing q are defined

$$\eta_1 = q + \sum \gamma_{2i-1} + \sum \text{ subarcs } t_{2i-1}s_{2i+1} \text{ of } \alpha,$$
$$\eta_2 = q + \text{subarc } s_1s_2 \text{ of } \alpha + \sum \gamma_{2i} + \sum \text{ subarcs } t_{2i}s_{2i+2} \text{ of } \alpha.$$

From step (d) the cyclic connectivity theorem follows easily. We have a simple closed curve containing q. Similarly one contains p. If these simple closed curves have two points in common, their sum contains a simple closed curve containing p and q. If they have but one point in common, their sum plus an arc pq not containing this point will contain the desired simple closed curve. If they do not intersect, then their sum plus an arc joining them plus its finite spanning system enables one to choose the desired two arcs. In this case the selection of the two arcs follows the methods used in picking η_1 and η_2 but the whole process here is finite.

Remarks. The cyclic connectivity theorem is a special case of the n-Bogensatz where $n = 2$ and the two closed sets are single points. All proofs that have been given for the n-Bogensatz are extremely long and intricate [**2**, **3**, **4**, **7**], and a simple proof of this important theorem would be a real contribution. Unfortunately the method used in this note does not appear to generalize to the higher values of n.

References

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[2] This arc would be given by q plus the subarc of β_1 from u_1 to the first point of β_2 on β_1, plus the subarc of β_2 from this point to the first point point of β_3, and so on. From diam $\beta_i \to 0$ it follows that this set is an arc with end points u_1 and q.

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INVERSES AND ZERO-DIVISORS

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It may happen that an element in a ring is both a zero-divisor and an inverse, that it possesses a right-inverse though no left-inverse, and that it is neither a zero-divisor nor an inverse. Thus there arises the problem of finding conditions assuring the absence of these paradoxical phenomena; and it is the object of the present note to show that chain conditions on the ideals serve this purpose. At the same time we obtain criteria for the existence of unit-elements.

The following notations shall be used throughout. The element e in the ring R is a *left-unit for the element* u in R, if $eu=u$; and e is a *left-unit for* R, if it is a left-unit for every element in R. Right-units are defined in a like manner; and an element is a *universal unit for* R, if it is both a right- and a left-unit for R.

The element u is a *right-zero-divisor*, if there exists an element $v\neq 0$ in R such that $vu=0$; and u is a *right-inverse in* R, if there exists an element w in R such that wu is a left-unit for u and a right-unit for R. Left-zero-divisors and left-inverses are defined in a like manner. Note that 0 is a zero-divisor, since we assume that the ring R is different from 0.

$L(u)$ denotes the set of all the elements x in R which satisfy $xu=0$; clearly $L(u)$ is a left-ideal in the ring R and every left-ideal of the form $L(u)$ shall be termed a *zero-dividing left-ideal. Principal left-ideals*[1] are the ideals of the form Rv for v in R and the ideals vR are the principal right-ideals.

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[1] This is a slight change from the customary terminology.