GENERALIZED FISCHER GROUPS AND ALGEBRAS[1]

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Introduction. In this paper we shall be concerned with the structure of the rational representation of certain sets of matrices, to which we give the name *generalized Fischer sets*. If K is any field, ϕ any fixed automorphism of K, and A any matrix with elements in K, we use the notation A^ϕ for the ϕ-automorph of A; that is, the matrix obtained from A by subjecting each of its elements to the automorphism ϕ. Again, we denote by A^* the transposed ϕ-automorph A'^ϕ of A.

DEFINITION. *A set $\mathfrak{M}$ of n-rowed square matrices which contains A^* if it contains A is defined to be a generalized Fischer set.*

Generalized Fischer groups, semi-groups and algebras are similarly defined.

In either of the special cases (1) $K=k$, a real field, ϕ is the identity automorphism; (2) $K=k^+=k(-1)^{1/2}$, ϕ is the operation of taking the conjugate complex, the set $\mathfrak{M}$ will be called a *Fischer set*. Fischer groups were probably named[2] by M. Schiffer, who, in 1933, proved in an unpublished work that every such group is completely reducible. This result has also been given by Specht [3], and will again be derived for all Kronecker product representations in the present paper (Theorem I, §4). In §1 we give a partial converse in the cases of the field of all reals (Example (6)), and the field of all complex numbers (Example (5)); this is summed up in Theorem II (§4).

Unlike Fischer sets, generalized Fischer sets and their rational representations are not always completely reducible; the regular representation of a finite group over a field of prime characteristic dividing the order of the group is a case in point (§1, Example (8)). When ϕ is non-involutary, the most we can give concerning the structure of g.F. sets is contained in Lemma II (§4) and Lemma IV (§5). But when ϕ is an involutary automorphism, a more satisfactory result is

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[1] The following is essentially contained in the author's doctorate thesis [1], written under the direction of Professor Richard Brauer. Professor Brauer has also offered many helpful suggestions in connection with the present paper. The thesis undertook a general study of $GL(n)$, and employed the results for specific calculation of the irreducible characters of $GL(4)$ over an infinite modular field.

[2] They were considered earlier by E. Fischer [2], who proved that the rational integral invariants of a Fischer group possessed a finite integrity basis.

given by Theorem III (§6); this states that each Kronecker product representation[3] of a g.F. set may be written in the form

$$\begin{pmatrix} \mathfrak{A} & & \\ & \mathfrak{B} & \\ * & & \mathfrak{C} \end{pmatrix}$$

where $\mathfrak{B}$ is in completely reduced form and $\mathfrak{C}$ is dual to $\mathfrak{A}$ in a sense which we will not describe at this point.

The main tool of the paper is the scalar product of two forms (or vectors) of the linear vector space S upon which the transformations of $\mathfrak{S}=\pi_f(\mathfrak{M})$ are performed. This is introduced in §2.

1. **Examples of generalized Fischer sets.** In this section proofs are omitted. The first two examples and the last hold true if the word set is replaced by group, semi-group or algebra. We use without specific mention material from [**4**], [**5**] and [**6**].

(1) *The direct sum of a finite number of g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(2) *The direct product of two g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(3) *A general linear group is a g.F. group w.r.t. every automorphsim ϕ of the underlying field.*

(4) *A total matric algebra is a g.F. algebra w.r.t. every ϕ.*

(5) *A semi-simple algebra of matrices over an algebraically closed field is similar to a g.F. algebra w.r.t. every ϕ.*

(6) *A semi-simple algebra of matrices over the field of all real numbers is similar to a Fischer algebra.*

(7) *Any group of matrices leaving invariant the bilinear form*

$$x^*y = \sum_{i=1}^{n} \overset{\phi}{x_i} y_i$$

is a g.F. group.

(8) *The right-hand (or first) regular representation $\mathfrak{R}$ of a finite group G is a g.F. group, and the linear closure of $\mathfrak{R}$ is a g.F. algebra, w.r.t. any ϕ, even if the underlying field is modular.*

(9) *If $\mathfrak{M}$ is a g.F. set w.r.t. the automorphism ϕ, its commutator algebra $\mathfrak{C}$ is a g.F. algebra w.r.t. the inverse automorphism $\theta=\phi^{-1}$.*

[3] By the fth Kronecker product representation of a set $\mathfrak{M}$ we mean simply the set $\mathfrak{S}=\pi_f(\mathfrak{M})$ consisting of the fth Kronecker product [**4**] of each matrix of $\mathfrak{M}$. If $\mathfrak{M}$ is a (semi-) group, $\mathfrak{S}$ forms a representation in the usual sense. The corresponding statement in the case of an algebra is true only if $f=1$; thus the paper gives information only about the algebra itself.

2. **Kronecker product representations.** The matrices of a g.F. set $\mathfrak{A}$ may be considered as transformations of a contravariant vector x^i

$$A:\quad x^{i} = \sum_{j=1}^{n} a_1^{j} x'^{j}. \tag{1}$$

Then the fth Kronecker product $\mathfrak{S} = \pi_f(\mathfrak{A})$ may be regarded as a set of transformations of the linear vector space S of all forms

$$\begin{aligned} F(T) &= \sum c_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \\ G(T) &= \sum d_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \end{aligned} \tag{2}$$

where $T^{i_1 i_2 \cdots i_f}$ is an arbitrary contravariant tensor with f indices. If under the transformation (1), T becomes T', then $F(T)$ becomes a new form $F_{(A)}(T')$ defined by

$$F(T) = F_{(A)}(T'). \tag{3}$$

We define the form $F_{(A)}(T)$ (in T, not T'!) to be the *transform of F by A*. Since

$$F_{(I)} = F, \qquad F_{(A)(B)} = F_{(AB)}, \tag{4}$$

it follows that if $\mathfrak{A}$ is a (semi-) group the set of transformations under $\mathfrak{A}$ of a basis of forms of S is a representation of $\mathfrak{A}$ similar to $\pi_f(\mathfrak{A})$.

We define the *scalar product* of the two forms F, G of (2) by

$$F \circ G = \sum c^{\theta}_{i_1 i_2 \cdots i_f} d_{i_1 i_2 \cdots i_f}, \qquad \theta = \phi^{-1}. \tag{5}$$

Clearly $F \circ G$ is a nondegenerate bilinear form in the two n^f-dimensional vectors F^θ, G whose coordinates are, respectively, $c^{\theta}_{i_1 i_2 \cdots i_f}$ and $d_{i_1 i_2 \cdots i_f}$. It has all the properties of such a bilinear form; we single out

$$(cF) \circ G = c^{\theta}(F \circ G), \qquad F \circ (cG) = c(F \circ G), \tag{6}$$

where c is any element of the underlying field. Again, it is readily verified that

$$F_{(A^*)} \circ G = F \circ G_{(A)}, \qquad A^* = A'^{\phi}. \tag{7}$$

These remarks and equations are fundamental to the present paper. Equation (7) explains the necessity for restricting attention to g.F. sets. Although all subsequent results are stated for the above Kronecker product representation they may be applied to others as well;[4] for this reason there would be some value in adopting a more axiomatic treatment. The essentials are: a vector space S with a g.F.

[4] In [1] the results were applied to Kronecker products of power products of alternating representations (that is, those afforded by skew-symmetric tensors) of $GL(n)$.

set $\mathfrak{A}$ of operators, as indicated by (4), and a nondegenerate scalar product $F \circ G$ bilinear in F^θ and G with the additional property (7).

We shall wish to refer to the following special properties which do not hold generally:

$$(H): \quad G \circ F = (F \circ G)^\theta, \qquad \phi = \theta. \tag{8}$$

This is true when ϕ is an involutary automorphism, and means that $F \circ G$ is a "hermitian" form.

$$(F): \quad G \neq 0 \quad \text{implies} \quad G \circ G \neq 0. \tag{9}$$

This is certainly true in each of the following cases: (1) $K = k$, a real field, ϕ is the identity automorphism; (2) $K = k^+$, ϕ is the operation of taking the conjugate complex. Thus (F), and indeed also (H), may be assumed in dealing with Fischer sets.

3. **Modules.** Any subspace $S_1 \subset S$ which is invariant under the g.F. set $\mathfrak{A}$ we will call a *module*. Let the module S_1 have basis F_i $(i = 1, 2, \cdots, \sigma_1)$, and define the transpose of the matrix M_1 by

$$M_1' = (F_i \circ F_j), \tag{10}$$

where i, j are, respectively, the row and column index of the σ_1-rowed square matrix on the right. M_1 will be referred to as *the matrix associated with the module* S_1. Its rank is invariant under change of basis of S_1. If M_1 is nonsingular, S_1 is a *nonsingular* module. If M_1 is the zero matrix, S_1 is a module *of rank zero*. Since S_1 is a module, $F_{i(A)}$ is in S_1:

$$F_{i(A)} = \sum_{j=1}^{\sigma_1} s_{ij}^{(1)}(A) \cdot F_j. \tag{11}$$

If we set

$$s_1(A) = (s_{ij}^{(1)}(A)) \tag{12}$$

then the set $\mathfrak{S}_1$ of matrices $s_1(A)$ forms a representation of $\mathfrak{A}$.

Definition. *Let $\mathfrak{A}$ be mapped upon a set $\mathfrak{U}$ of rectangular matrices: $A \to u(A)$. Then $\overline{\mathfrak{U}}$ denotes the set of matrices $\bar{u}(A) = [u(A^*)]'^\theta$.*

Note that if $\phi = \theta$ is involutary, $\bar{u}(A) = u^*(A^*)$. Moreover if $\mathfrak{R}$ is a representation of a semi-group $\mathfrak{G}$, then $\overline{\mathfrak{R}}$ is also a representation.

Lemma I. *With the above notations*

$$M_1 \cdot \overline{\mathfrak{S}}_1 = \mathfrak{S}_1 \cdot M_1. \tag{13}$$

PROOF. By (7)

$$F_{j(A^*)} \circ F_i = F_j \circ F_{i(A)},$$

and by (6)

$$\sum_k F_k \circ F_i \cdot s_{jk}^{(1)\theta}(A^*) = \sum_l s_{il}^{(1)}(A) \cdot F_j \circ F_l.$$

But in matrix notation, if i, j are taken as row and column index, respectively, the last equation is identical with (13).

LEMMA II. *An irreducible module is either nonsingular or of rank zero.*

PROOF. Let S_1, M_1, $\mathfrak{S}_1$ be, respectively, the irreducible module, its associated matrix and the corresponding (irreducible) set of matrices. From Schur's lemma [5] and equation (13), M_1 is either nonsingular or zero.

We restrict our attention to modules of the two types mentioned in Lemma II, without, however, requiring them to be irreducible. The full space S is itself a nonsingular module. In fact, since the bilinear form $F \circ G$ is nondegenerate, the equation $F \circ G = 0$ holding for a fixed G and all F of S implies $G = 0$; it follows readily that the matrix M associated with S is nonsingular.

4. **Nonsingular modules.** Let the module S_1 be a proper subspace of S, in the sense that not every form of S is in S_1.

LEMMA III. *If S_1 is a nonsingular module, then S is the direct sum of S_1 and another nonsingular module S_2, and $\mathfrak{S}$ is decomposable:*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix}. \tag{14}$$

PROOF. Let F_i $(i = 1, 2, \cdots, \sigma_1)$ be a basis of S_1, and let these with G_α $(\alpha = \sigma_1 + 1, \cdots, \sigma)$ form a basis of S. Let G be any form of S. Then since the matrix M associated with S is nonsingular, we may uniquely determine elements $u_1, u_2, \cdots, u_{\sigma_1}$ such that

$$F = u_1 F_1 + u_2 F_2 + \cdots + u_{\sigma_1} F_{\sigma_1}$$

satisfies $F_i \circ F = F_i \circ G$, all i. Thus the form $G' = G - F$ satisfies $F_i \circ G' = 0$, for all i. Therefore, without loss of generality we may assume that the equations

$$F_i \circ G_\alpha = 0 \tag{15}$$

hold for all i, α. The matrix M associated with S then takes the form

$$M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix} \tag{16}$$

where $M_2 = (G_\alpha \circ G_\beta)'$ is, with M and M_1, nonsingular. Since S_1 is a module, $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are given by

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ \mathfrak{U} & \mathfrak{S}_2 \end{pmatrix}, \qquad \overline{\mathfrak{S}} = \begin{pmatrix} \overline{\mathfrak{S}}_1 & \overline{\mathfrak{U}} \\ 0 & \overline{\mathfrak{S}}_2 \end{pmatrix}. \tag{17}$$

But Lemma I, quoted for S, takes the form $M\overline{\mathfrak{S}} = \mathfrak{S}M$. From (16) and (17) follows $0 = \mathfrak{U}M_1$, $\mathfrak{U} = 0$. Thus $\mathfrak{S}$ assumes the form (14), and the set of all linear combinations of the G_α is a module; nonsingular, since M_2 is nonsingular.

When property (F) holds (see equation (9)), S clearly possesses no modules of rank zero. Hence we have this lemma.

LEMMA IIIa. *If (F) is true in S, $\mathfrak{S}$ is completely reducible.*

As an immediate consequence we have the following theorem.

THEOREM I. *The Kronecker product representations of a Fischer set are completely reducible.*

In particular the theorem states that Fischer groups and algebras are completely reducible. As a partial converse we have from examples (6) and (5) of §1 that a semi-simple algebra over the field of all real numbers or the field of all complex numbers is similar to a Fischer algebra.

THEOREM II. *Let $\mathfrak{G}$ be a (semi-) group of matrices over k or $k(-1)^{1/2}$, (k the field of all real numbers). Let $\mathfrak{A}$ be the linear closure of $\mathfrak{G}$. Then a necessary and sufficient condition that $\mathfrak{G}$ (and its Kronecker product representations) be completely reducible is that $\mathfrak{A}$ be similar to a Fischer algebra.*

5. **Modules of rank zero.** We now prove this lemma.

LEMMA IV. *If S_1 is a module of rank zero, then, in the sense of similarity,*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & & \\ & \mathfrak{S}_2 & \\ * & & \overline{\mathfrak{S}}_1 \end{pmatrix}. \tag{18}$$

PROOF. Let F_i $(i = 1, 2, \cdots, \sigma_1)$, be a basis of S_1, and let G_α $(\alpha = \sigma_1 + 1, \cdots, \sigma)$ complete the basis of S. Consider the matrix

$$U = (F_i \circ G_\alpha) \tag{19}$$

and the form $F = v_1F_1 + v_2F_2 + \cdots + v_{\sigma_1}F_{\sigma_1}$. If the matrix U had rank less than σ_1, we could choose a set of v's, not all zero, so that $F \circ G_\alpha = 0$ for all α. This, coupled with the fact that $F \circ F_i = 0$ for all i (since S_1 is a module of rank zero), would imply $F \circ G = 0$ for every G of S, in contradiction to the fact that the bilinear form $F \circ G$ is nondegenerate. Thus U has rank σ_1, and there exist (cf. [7]), two nonsingular matrices $P = (p_{ij})$ and $Q = (q_{\alpha\beta})$ such that $PUQ = (0 | I_{\sigma_1})$. We make the transformation of basis given by

$$\overline{F}_i = \sum p_{ij}^{\phi} F_j, \qquad \overline{G}_\alpha = \sum q_{\beta\alpha} G_\beta \tag{20}$$

so that

$$\overline{U} = (\overline{F}_i \circ \overline{G}_\alpha) = PUQ = (0 | I_{\sigma_1}). \tag{21}$$

Without loss of generality we may assume that (21) holds for U. We rename the last basis elements, calling them H_i ($i = 1, 2, \cdots, \sigma_1$), so that our basis is now

$$F_1, F_2, \cdots, F_{\sigma_1}, G_{\sigma_1+1}, \cdots, G_{\sigma-\sigma_1}, H_1, H_2, H_{\sigma_1},$$

where the G's now appear only when $\sigma - 2\sigma_1 > 0$, but the H's necessarily appear. Calculation of M for this basis gives, in view of (21),

$$M = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ I_{\sigma_1} & * & * \end{pmatrix}. \tag{22}$$

We may assume

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 & 0 \\ * & \mathfrak{S}_2 & \mathfrak{U} \\ * & * & \mathfrak{S}_3 \end{pmatrix}, \qquad \overline{\mathfrak{S}} = \begin{pmatrix} \overline{\mathfrak{S}}_1 & * & * \\ 0 & \overline{\mathfrak{S}}_2 & * \\ 0 & \overline{\mathfrak{U}} & \overline{\mathfrak{S}}_3 \end{pmatrix}. \tag{23}$$

Using Lemma I, or $M\overline{\mathfrak{S}} = \mathfrak{S}M$, we obtain: by comparing the blocks (2, 1), $\mathfrak{U} = 0$; by comparing the blocks (3, 1), $\mathfrak{S}_3 = \overline{\mathfrak{S}}_1$. This completes the proof.

Lemma IV seems to be the most we can say when ϕ is a non-involutary automorphism.

Assume that $\phi = \theta$ is involutary, so that property (H) (equation (8)) holds for the scalar product. In this case (22) may be replaced by

$$M = \begin{pmatrix} 0 & 0 & I_{\sigma_1} \\ 0 & N & * \\ I_{\sigma_1} & * & * \end{pmatrix}, \qquad N = (G_h \circ G_k)', \tag{22a}$$

where h and k have the range σ_1+1 to $\sigma-\sigma_1$. The set S_2 of all linear combinations of the G_h is a linear vector space. For S_2 we may obtain a useful scalar product of two forms

$$G = x_{\sigma_1}G_{\sigma_1} + \cdots + x_{\sigma-\sigma_1}G_{\sigma-\sigma_1}, \quad R = y_{\sigma_1}G_{\sigma_1} + \cdots + y_{\sigma-\sigma_1}G_{\sigma-\sigma_1}, \tag{24}$$

by merely taking over the scalar product for S

$$G \circ R = \sum n_{kh} x_h^{\phi} y_k, \qquad n_{kh} = G_h \circ G_k. \tag{25}$$

From (22a), det $M = \pm$det $N \neq 0$, hence $N = (n_{hk})$ is nonsingular, $G \circ R$ is nondegenerate. In order to obtain a property comparable to (7) we must redefine the transform of a form of S_2 by a matrix of $\mathfrak{A}$. Since, by Lemma IV, S_2 is invariant (modulo S_1) under $\mathfrak{A}$,

$$G_{(A)} = G(A) + F(A) \tag{26}$$

where $G(A)$, $F(A)$ are uniquely determined forms of S_2, S_1. We define $G(A)$ to be the transform (*in* S_2) of G by A. Then, by virtue of (22a),

$$G(A^*) \circ R = G_{(A^*)} \circ R = G \circ R_{(A)} = G \circ R(A). \tag{27}$$

Formula (27) gives the desired property. There results this lemma.

LEMMA IVa. *If the automorphism ϕ is involutary, the representation $\mathfrak{S}_2$ of Lemma* IV *may be associated with a vector space S_2, with operator-set $\mathfrak{A}$, which possesses a nondegenerate* (*hermitian*) *scalar product satisfying* (7). *Thus the preceding methods and results may be applied to S_2 and $\mathfrak{S}_2$ just as they were to S and $\mathfrak{S}$.*

6. **The structure of** S. Lemmas III and IV may be combined in a number of ways to give information concerning the structure of S and $\mathfrak{S}$. This information is annoyingly limited, however, in case we cannot assume the conclusion of Lemma IVa. The following theorem therefore concerns only g.F. sets defined relative to an involutary automorphism.

THEOREM III. *Let $\mathfrak{A}$ be a set of n-rowed square matrices which contains $A^* = A'^{\theta}$ with every A, where θ is a fixed involutary automorphism of the underlying field. If $\mathfrak{R} = \{R(A)\}$ is any representation of $\mathfrak{A}$, define*

$$\overline{\mathfrak{R}} = \{\overline{R}(A)\},$$

where

$$\overline{R}(A) = [R(A^*)]^*.$$

Then the fth Kronecker product representation $\pi_f(\mathfrak{A})$ of $\mathfrak{A}$ is given in reduced form by

$$(28)\qquad \pi_f(\mathfrak{A}) = \left(\begin{array}{c|c|c} \begin{matrix}\mathfrak{B}_1 & & & \\ & \mathfrak{B}_2 & & \\ & & \ddots & \\ & * & & \mathfrak{B}_\kappa\end{matrix} & & \\ \hline * & \begin{matrix}\mathfrak{C}_1 & & & \\ & \mathfrak{C}_2 & & \\ & & \ddots & \\ & 0 & & \mathfrak{C}_\lambda\end{matrix} & \\ \hline * & * & \begin{matrix}\bar{\mathfrak{B}}_\kappa & & & \\ & \ddots & & \\ & & \bar{\mathfrak{B}}_2 & \\ & * & & \bar{\mathfrak{B}}_1\end{matrix}\end{array}\right),$$

where the $\mathfrak{B}$*'s and* $\mathfrak{C}$*'s are irreducible representations, and each* $*$ *denotes elements not necessarily zero.*

The wording of the theorem is especially adapted to the case that $\mathfrak{A}$ is a (semi-) group. If $\mathfrak{A}$ is an algebra, the last sentence should be replaced by "Then $\mathfrak{A}$ is given in reduced form by $\cdots$."

PROOF. First we apply Lemma IV a number of times, assuming in each case that the S_1 is an irreducible module. After a finite number of steps, say κ, we must reach a projection space which contains no modules of rank zero. On applying Lemma III a number of times to this projection space we obtain the completely reduced block of $\mathfrak{C}$'s.

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