MEASURE AND OTHER PROPERTIES OF A HAMEL BASIS

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A Hamel basis[1] is a set $a, b, c, \cdots$ of real numbers such that if x is any real number whatsoever then x may be expressed uniquely in the form $\alpha a+\beta b+\gamma c+\cdots$ where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Since each of these sums is formed from a finite number of nonzero terms and the coefficients $\alpha, \beta, \gamma, \cdots$ are rational and therefore form a countable set, it seems intuitively plausible that not only should the basis set be of the same power as the continuum but in some way be of the same "thickness" as the continuum. However, this intuitive feeling is seemingly contradicted by the only known results along this line, namely: the inner measure of a Hamel basis is zero and its outer measure may also be zero.[2] Nevertheless, this intuition is justified to some extent by Theorems 2, 4, and 5. A natural question arises: In order for a set of real numbers to contain a Hamel basis, what is both necessary and sufficient? For a certain family of sets (including the Borel and analytical sets) this question is answered in two ways. Certain other properties of a Hamel basis are investigated, the most interesting being an example of a Hamel basis which contains a nonvacuous perfect set. Finally, some rather curious discontinuous solutions of the equation $f(x)+f(y)=f(x+y)$ are given.

Measure. No Hamel basis of positive exterior measure is measurable.[3] The next few theorems show this to be true also of certain transforms of every Hamel basis.

DEFINITION. *If M is a set of real numbers, by $T(M)$ is meant the set of all numbers x' such that $x'=x+(y'-y)$, where x, y, and y' belong to M.*

With M considered as a linear set, $T(M)$ is the sum of all translations of M which intersect M. For convenience, $T[T(M)]$ is abbreviated $T^2(M)$, $T[T^2(M)]$ is abbreviated $T^3(M)$, and so on and $T^0(M)=M$.

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[1] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y)=f(x)+f(y)$*, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[2] W. Sierpinski, *Sur la question de la mesurabilité de la base de M. Hamel*, Fundamenta Mathematicae, vol. 1 (1920), pp. 105–111.

[3] Ibid.

THEOREM 1. *If H is a Hamel basis, then for each positive integer n, $m_iT^n(H)=0$.*[4]

PROOF. Suppose, on the contrary, that for some positive integer n, $m_iT^{n-1}(H)>0$. Let a and b denote two numbers of H. Since the set of distances of the set $T^{n-1}(H)$ contains an entire interval with its left end at zero,[5] $T^n(H)$ contains an interval V whose midpoint is b. Since a/b is irrational and a is not zero, there exists a rational number r which is not an integer such that ra belongs to V and hence to $T^n(H)$. But every number of $T^n(H)$ can be expressed in the form $n_1x_1+n_2x_2+n_3x_3+\cdots$ where $x_1, x_2, x_3, \cdots$ belong to H and $n_1, n_2, n_3, \cdots$ are *integers* of which only a finite number are different from zero. This is contrary to the properties of H.

THEOREM 2. *If H is a Hamel basis, then for some positive integer n, $m_eT^n(H)>0$.*

PROOF. Suppose, on the contrary, that for each positive integer n, $mT^n(H)=0$. Hence $m\sum_0^\infty T^n(H)=0$. Let M_0 denote $\sum T^n(H)$, and let a denote a fixed number of M_0. Evidently every number of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots$ (where $b, c, d, \cdots$ belong to H and $n_1, n_2, n_3, \cdots$ are integers of which only a finite number are different from zero) belongs to M_0. Hence the set of all numbers of this form are of measure zero. So the set of all numbers of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots+n_0a$ (where n_0 is an integer and all other symbols have the same meaning as before) is of measure zero. Now any real number x can be expressed in the form $\alpha a+\beta b+\gamma c+\cdots$ (where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero) and hence there exists an integer q (which depends on x) such that $q\alpha, q\beta, q\gamma, \cdots$ are integers of which only a finite number are different from zero. Consequently, qx is of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots+n_0a$. It follows that the set of all real numbers x is of measure zero, which is false.

DEFINITION. *If M is a set of real numbers, then $D(M)$ denotes the set of all numbers $x-y$ where x and y belong to M and $x\geqq y$. The set $D(M)$ is called the set of distances of the set M.*[6]

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[4] Throughout this paper, if Q is a set, m_iQ and m_eQ denote the interior measure and exterior measure, respectively, of Q and mQ denotes the measure of Q if measurable (in the sense of Lebesgue).

[5] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fundamenta Mathematicae, vol. 1 (1920), pp. 93–104.

[6] Steinhaus, loc. cit.

THEOREM 3. *If H is a Hamel basis, then for each positive integer n, $m_iD^n(H)=0$.*

Theorem 3 may be established by substituting D for T in the proof of Theorem 1.

THEOREM 4. *If H is a Hamel basis, then for some positive integer n, $m_eD^n(H)>0$.*

PROOF. Suppose the contrary. Let $N=\sum_1^\infty D^n(H)$. Then $mN=0$. Let Q denote the set of all points of H which are limit points of H from both sides, and let M denote a countable subset of H such that (1) M contains $H-Q$ and (2) every point of $H-M$ is a limit point of M. It follows that each point of $H-M$ is a limit point of M from both sides. Let x denote a real number. Then x is of the form $\alpha a+\beta b+\gamma c+\cdots$ (where $a, b, c, \cdots$ belong to H and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero). Hence there exists an integer q such that qx is of the form $n_1a+n_2b+n_3c+\cdots$ (where $n_1, n_2, n_3, \cdots$ are integers of which only a finite number are different from zero). So qx is of the form $\sum_1^j e_ix_i$ (where for each i, $i=1, 2, \cdots, j$, x_i belongs to H and e_i is ± 1). For each integer i, $i=1, 2, \cdots, j$ (j is finite and depends on x), there exists a number y_i of M such that $\sum_1^j e_i(x_i-y_i)$ belongs to N.[7] Hence $qx=\bar{x}+\sum_1^j e_iy_i$ where $\bar{x}$ belongs to N. Since there are only countably many numbers of the form $\sum_1^j e_iy_i$, it follows that the set of all real numbers x is of measure zero, which is a contradiction.

LEMMA 1. *If M is a set of numbers and every number of some number interval V can be expressed in the form $\alpha a+\beta b+\gamma c+\cdots$ (where $a, b, c, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero), then M contains a Hamel basis.*

Lemma 1 may be established by well-ordering M and applying Hamel's argument to this well-ordering.

THEOREM 5. *If M is a set of real numbers and for some positive integer n, either $m_iT^n(M)>0$ or $m_iD^n(M)>0$, then M contains a Hamel basis.*

Theorem 5 follows from Lemma 1 and one of Steinhaus' theorems on the set of distances of a set.[8] The condition in Theorem 5 under

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[7] This can be done by first choosing y_1 so that $e_1(x_1-y_1)\geqq 0$. This makes $e_1(x_1-y_1)$ belong to $D(H)$. If it is zero, choose y_2 so that $e_2(x_2-y_2)\geqq 0$; but if it is positive choose y_2 so that $e_2(x_2-y_2)\leqq 0$ but $e_1(x_1-y_1)+e_2(x_2-y_2)\geqq 0$. This makes $e_1(x_1-y_1)+e_2(x_2-y_2)$ belong to $D^2(H)$. Continue this process.

[8] Loc. cit., Theorem 8, p. 99.

which the set M contains a Hamel basis is not necessary. This can be seen from Theorems 1 and 3.

EXAMPLE 0. *The Cantor discontinuum*[9] *contains a Hamel basis, because its set of distances contains the interval from zero to one.*

DEFINITIONS. *Let F_T denote the family of all sets M of real numbers such that for infinitely many different positive integral values of n, $T^n(M)$ is measurable; and let F_D denote the family of all sets M of real numbers such that for infinitely many different positive integral values of n, $D^n(M)$ is measurable.*

THEOREM 6. *If M is a set of the family F_T, then in order that M contain a Hamel basis it is necessary and sufficient that for some positive integer n, $mT^n(M) > 0$.*

THEOREM 7. *If M is a set of the family F_D, then in order that M contain a Hamel basis it is necessary and sufficient that for some positive integer n, $mD^n(M) > 0$.*

LEMMA 2. *If M is an analytical set of real numbers, then both $T(M)$ and $D(M)$ are analytical sets.*

PROOF. Being an analytical set of real numbers, M is the set of values of a function $f_1(x)$ of a real variable, defined and continuous in the set of all irrational numbers.[10] Furthermore, $D(M)$ is analytical[11] and the set N consisting of all real numbers x such that either x or $-x$ belongs to $D(M)$ is an analytical set. Hence N is the set of values of a function $f_2(x)$ of a real variable, defined and continuous in the set of all irrational numbers. Let $f(x, y) \equiv f_1(x) + f_2(y)$ where x and y are irrational numbers. Then $T(M)$ is the set of values of $f(x, y)$ and $f(x, y)$ is defined and continuous in the set of all points of the number plane whose coordinates are both irrational numbers. It follows that $T(M)$ is analytical.[12]

Perfect sets, analytical sets. Since every analytical set is measurable,[13] the following three theorems may be easily established using Lemma 2 and Theorems 6 and 7.

[9] The subset of the interval from 0 to 1 (of real numbers) remaining after deleting in succession every middle-1/3 segment (open interval).

[10] W. Sierpinski, *Introduction to General Topology*, The University of Toronto Press, Toronto, 1930, translated by C. C. Krieger, Theorem 73, p. 145.

[11] W. Sierpinski, *Sur l'ensemble de distances entre les points d'un ensemble*, Fundamenta Mathematicae, vol. 7 (1925), pp. 144–148, p. 146 in particular.

[12] W. Sierpinski, *General Topology*, loc. cit., Theorem 89, p. 185.

[13] N. Lusin, *Leçons sur les ensembles analytiques*, Gauthier-Villars, Paris, 1930, p. 152.

THEOREM 8. *In order that an analytical set A of real numbers shall not contain a Hamel basis it is necessary and sufficient that for each positive integer n, both $mT^n(A)$ and $mD^n(A)$ be zero.*

THEOREM 9. *No Hamel basis is an analytical set.*

THEOREM 10. *If an analytical set M is a subset of a Hamel basis, then for each positive integer n, both $mT^n(M)$ and $mD^n(M)$ are zero.*

THEOREM 11. *Suppose that $f(x)$ is a discontinuous real function which is defined for all real values of x and which satisfies the equation $f(x)+f(y)=f(x+y)$. If $f(x)$ is bounded over the analytical set K of real numbers, then for each positive integer n, $mT^n(K)=0$.*

PROOF. Since $f(x)$ is bounded over K, there exists a positive number B such that $|f(x)|<B$ if x belongs to K. It follows from the elementary properties of $f(x)$ (which are imposed upon it by the functional equation) that for any positive integer n, $|f(x)|<3^nB$ if x belongs to $T^n(K)$. Suppose that for some integer $\bar{n}$, $mT^{\bar{n}}(K)\neq 0$. Since $T^{\bar{n}}(K)$ is analytical, it is measurable, and hence, is of *positive* measure. Then $T^{\bar{n}+1}(K)$ contains an interval. Since $f(x)$ is bounded over this interval, by one of Darboux's theorems it must be continuous.[14] This is a contradiction.

COROLLARY 1. *If the discontinuous solution $f(x)$ of the functional equation $f(x)+f(y)=f(x+y)$ is continuous in an analytical set K of real numbers, then for each positive integer n, $mT^n(K)=0$.*

COROLLARY 2. *Suppose that A is an analytical subset of the plane image of a discontinuous solution of the equation $f(x)+f(y)=f(x+y)$ and that K is the projection of A onto the x-axis. Then for each positive integer n, $mT^n(K)=0$.*

THEOREM 12. *No discontinuous solution of the equation $f(x)+f(y)=f(x+y)$ is continuous in an analytical set which contains a Hamel basis.*

Theorem 12 follows immediately from Theorem 8 and Corollary 1.

Remarks and examples. Burstin showed the existence of a Hamel basis H which intersects every perfect set of real numbers.[15] It fol-

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[14] Darboux, *Sur la composition des forces in statique*, Bulletin des Sciences Mathématiques, vol. 9 (1875), p. 281.

[15] C. Burstin, *Die Spaltung des Kontinuum in c in L Sinne nichtmessbare Mengen*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, Mathematisch-naturwissenschaftliche Klasse, Abt. IIa, vol. 125 (1916).

lows that H does not contain a perfect set. But every uncountable analytical set contains a perfect set.[16] Consequently, this particular Hamel basis H does not contain an uncountable analytical set. This example and Theorems 9 and 10 might lead one to conjecture that no Hamel basis whatsoever contains an uncountable analytical set. That this conjecture would be false is shown by the following example.

EXAMPLE 1. *There exists a Hamel basis which contains a perfect set.*[17]

CONSTRUCTION. Let $r_1, r_2, r_3, \cdots$ denote a simple well-ordering of the rational numbers such that $r_1=0$. Let I_{11} denote a closed interval of real numbers not containing zero. The interval I_{11} contains two intervals I_{21} and I_{22} such that (1) I_{21} precedes I_{22} and (2) no number of the form $n_1x_1+n_2x_2$ except the forms x_1 or x_2 belongs to $I_{21}+I_{22}$, where $x_1, x_2 \in I_{21}+I_{22}$ and where $n_i=r_j$, $i, j=1$ or 2. Likewise I_{21} contains two intervals I_{31} and I_{32}, and I_{22} contains two intervals I_{33} and I_{34} such that (1) I_{31} and I_{33} precede I_{32} and I_{34}, respectively, and (2) no number of the form $n_1x_1+n_2x_2+n_3x_3$ except the forms x_1, x_2, or x_3 belongs to $\sum I_{3n}$ where $x_1, x_2, x_3 \in \sum I_{3n}$ and where $n_i=r_j$, $i, j=1, 2$, or 3. This process may be continued. For each positive integer k, let G_k denote the collection of mutually exclusive intervals $I_{k1}, I_{k2}, \cdots, I_{kq}$, where $q=2^{k-1}$. For each k, each element of G_k contains two elements of G_{k+1} and if $x_1, x_2, \cdots, x_k$ denote numbers of G_k^*, then no number of the form $\sum_1^k n_ix_i$ except the form x_i belongs to G_k^*, where $n_i=r_j$, $i, j=1, 2, \cdots, k$.[18] Now let M denote $\prod_1^\infty G_i^*$. Evidently M is a perfect set such that if $\bar{x}$ is a number of M, then $\bar{x} \neq \alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ are numbers of $M-\bar{x}$ and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. It follows from this that if $\bar{x}$ is a real number which can be expressed in the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero, then it can be thus expressed in only one way, that is, the form is unique.

Let Γ denote a well-ordering of the real numbers not belonging to M and let a denote the first number of Γ which is not of the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from

[16] Lusin, loc. cit., p. 151.

[17] R. L. Swain, in a conversation with me, demonstrated the existence of a perfect set whose set of distances contains no rational number. His method of construction forms the kernel of the one that I use here.

[18] G_k^* denotes the sum of the elements of G_k.

zero. Let b denote the first number of Γ which is not of the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to $M+a$. This process may be continued. It follows from Hamel's argument that $M+a+b+c+\cdots$ is a Hamel basis.[19]

Since, from Theorem 11, no discontinuous (real) solution of the functional equation $f(x)+f(y)=f(x+y)$ can be continuous at any point or continuous in any set of positive measure, could such a discontinuous solution be continuous in some *perfect* set? And could its image in the number plane (the graph of $y=f(x)$) at the same time be *connected*? Example 2 shows that the answer is *yes*.

EXAMPLE 2. *There exist a discontinuous (real) solution of the functional equation $f(x)+f(y)=f(x+y)$ and a perfect set M of real numbers such that* (1) $f(x)=0$ *if x belongs to the perfect set M and* (2) *the plane image of $f(x)$ is connected.*

CONSTRUCTION. Let M denote a perfect subset of a Hamel basis H such that $H-M$ contains a perfect set. For each number x of M, let $f(x)$ be defined to be zero. Define the function $f(x)$ for each of the numbers of $H-M$ in such a way so that if Q is a continuum in the number plane not lying wholly in a vertical line, then for some number x of $H-M$, the point $(rx, rf(x))$ belongs to Q.[20] Now if x is any number not belonging to H, $x=\alpha a+\beta b+\gamma c+\cdots$ where $a, b, c, \cdots$ belong to H and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Let $f(x)=\alpha f(a)+\beta f(b)+\gamma f(c)+\cdots$. It follows from Hamel's argument that $f(x)$ satisfies the functional equation.[21] Furthermore, $f(x)$ has the following properties: (0) $f(x)$ is totally discontinuous but (1) $f(x)$ is zero for all numbers in the perfect set M and (2) the plane image of $f(x)$ is connected.[22]

In order for the plane image I of *any* discontinuous solution of $f(x)+f(y)=f(x+y)$ to be connected, I must intersect every[23] continuum in the plane which does not lie in some vertical line. However, I need not intersect every perfect subset of the plane not lying in the sum of a countable number of vertical lines, for the image of $f(x)$ in Example 2 does not contain any point of the line $y=1$ whose abscissa belongs to M. So by virtue of the fact that I may be connected and

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[19] G. Hamel, loc. cit. $M+a+b+c+\cdots$ denotes the sum of the *sets*, $M, a, b, c, \cdots$.

[20] F B. Jones, *Connected and disconnected plane sets and the functional equation* $f(x)+f(y)=f(x+y)$, this Bulletin, vol. 48 (1942), pp. 115–120, Theorems 2 and 4; r is rational.

[21] G. Hamel, loc. cit

[22] F. B. Jones, loc. cit.

[23] Ibid., Theorem 2.

still *contain* a perfect set not lying in the sum of countably many vertical lines, it follows that I need not *intersect* every such perfect set. Since, by Theorem 11, Corollary 2, I can *not* contain a perfect set whose projection on the x-axis is of positive measure, one might suppose that I must intersect every such perfect set in order to be connected. The following example shows this supposition to be false.

EXAMPLE 3. *There exists a discontinuous solution of the equation* $f(x)+f(y)=f(x+y)$ *such that* (1) *the image I of $f(x)$ in the number plane E is connected but* (2) *the intersection of I with the x-axis is exactly the set of rational numbers.*

Two lemmas are necessary.

LEMMA 3. *Suppose that M is a bounded, closed subset of the number plane. If uncountably many horizontal lines intersect M in an uncountable set, then each of c horizontal lines intersects M in an uncountable set.*[24]

PROOF. Let E denote the number plane. Let W denote the set of all points w of the y-axis for which there exist sets M_w such that (1) M_w is a subset both of a horizontal line in E through w and of M and (2) every point of M_w is a point of condensation of M_w from both sides. There exists a pair of vertical lines L_{11} and L_{12} (L_{11} being to the left of L_{12}) and an uncountable subset W_1 of W such that for each element w of W_1, M_w contains points between L_{11} and L_{12}, to the left of L_{11} and to the right of L_{12}. Let P_{11} denote a point of M lying between L_{11} and L_{12} such that for uncountably many different elements w of W_1, M_w is a subset of a horizontal line lying above P_{11} in E and for uncountably many elements w of W_1, M_w is a subset of a horizontal line lying below P_{11} in E. Let W_1^+ and W_1^- denote the set of all of those elements w of W_1 for which M_w is on a line above P_{11} and below P_{11}, respectively. Select one of the sets W_1^+ and W_1^- and denote the selection by $W_1^{\pm}$. Then there exist vertical lines L_{21}, L_{22}, L_{23}, and L_{24} having that order from left to right (with P_{11} between L_{22} and L_{23}) and an uncountable subset W_2 of $W_1^{\pm}$ such that for each w of W_2, M_w has points to the left of L_{21}, between L_{21} and L_{22}, between L_{22} and the vertical line through P_{11}, between this line and L_{23}, between L_{23} and L_{24}, and to the right of L_{24}. Let P_{21} and P_{23} denote two points of M which lie on the same horizontal line such that uncountably

[24] c is the cardinal number of the continuum. It would follow from this lemma that any closed subset of the plane which cannot be covered by a countable collection of horizontal and vertical lines can be covered only by a collection of horizontal and vertical lines which contains at least c lines.

many elements of W_2 lie above and uncountably many elements of W_2 lie below this line, denoting them by W_2^+ and W_2^-. Select *one* of the sets W_2^+ and W_2^- and denote the selection by $W_2^{\pm}$. This process may be continued and an infinite sequence of points P_{11}, P_{21}, P_{23}, P_{31}, P_{33}, P_{35}, P_{37}, $\cdots$ obtained whose limiting set Q is uncountable and lies both in M and in a horizontal line. Since for each n, $n=2, 3, 4, \cdots$ there are *two* selections possible in the selection of $W_n^{\pm}$, there are $2^{\aleph_0}$ such limiting sets of which no three are identical. Hence there are c different such sets Q.

LEMMA 4. *No compact continuum M lying in the number plane which is not wholly in a vertical line and whose common part with every horizontal line is totally disconnected is a subset of the sum of less than c vertical and horizontal lines.*

Lemma 4 may be established by an indirect argument with the help of Lemma 3 and the fact that an uncountable closed plane set contains c points.

CONSTRUCTION OF EXAMPLE 3. Let Γ denote a well-ordering of type Ω_0 (Ω_0 is the smallest ordinal having c ordinals less than it) of the set of all nondegenerate compact subcontinua of E not lying in a vertical line and let $x_1, x_2, x_3, \cdots, x_\omega, x_{\omega+1}, \cdots, x_z, \cdots, z<\Omega_0$, denote a well-ordering of a Hamel basis H such that $x_1=1$ and H contains a number of every perfect set of real numbers. Let $y_1=f(x_1)=f(1)=0$. If r is a rational number, let $f(r)=rf(1)=0$. Suppose that, for each ordinal $z<z_0<\Omega_0, f(x_z)$ is defined to be a number y_z. Then if $x=\sum r_z x_z$, $z<z_0$, $f(x)=\sum r_z y_z$, $z<z_0$, where r_z is a rational number and *not* zero for only a finite number of different values of z. Let I_{z_0} denote the image of $f(z)$ as defined so far. Let M_{z_0} denote the first element of Γ which contains no point of I_{z_0}. Let z_1 denote the smallest ordinal such that (1) $f(x_{z_1})$ is not defined so far and (2) there exists a number y_{z_1} such that (x_{z_1}, y_{z_1}) belongs to M_{z_0} and y_{z_1} is not the ordinate of any point of I_{z_0}. Since $z<\Omega_0$, the existence of z_1 may be established with the help of Lemma 4 and the theorem that every uncountable inner-limiting set of real numbers contains c mutually exclusive perfect sets. Then let $f(x_{z_1})=y_{z_1}$. If $\bar{z}$ is an ordinal less than z_1 such that for each ordinal $z<\bar{z}, f(x_z)$ is defined to be y_z, then let $f(x_{\bar{z}})$ be a real number $y_{\bar{z}}$ such that $y_{\bar{z}}$ is not $r_{z_1}y_{z_1}+\sum r_z y_z$, $z<\bar{z}$, where r_z is a rational number which is different from zero for only a finite number of different values of z and r_{z_1} is a rational number.

This completes the induction in the definition of $f(x)$ if x belongs to H. If x is any real number, then $x=\sum r_z x_z$, where r_z is a rational number which is different from zero for only a finite number of different values of z. This expression for x being unique, $f(x)$ is defined to be $\sum r_z y_z$. By Hamel's argument[25] $f(x)$ satisfies the equation $f(x)+f(y) = f(x+y)$. Obviously the image I of $f(x)$ intersects every compact subcontinuum of E not in a vertical line and hence intersects every such subcontinuum of E whether compact or not. Hence I is connected.[26] Suppose that for two different numbers a_1 and a_2, $f(a_1) = f(a_2)$. But $a_1 = \sum r_{1z} x_z$ and $a_2 = \sum r_{2z} x_z$, where r_{1z} and r_{2z} are rational numbers which are different from zero for only a finite number of different values of z. Hence $\sum r_{1z} y_z = \sum r_{2z} y_z$. Let $\bar{z}$ be the largest value of z such that $r_{1z} - r_{2z} \neq 0$. Then $y_z = \sum y_z (r_{2z} - r_{1z})/(r_{1\bar{z}} - r_{2\bar{z}})$, $z < \bar{z}$. This however is impossible if $\bar{z} > 1$. Hence $r_{1z} - r_{2z} = 0$ if $z > 1$ and $a_1 = a_2 = r_{11} - r_{21}$. It follows from this that if L is a horizontal line intersecting I at the point (x, y) then $I \cdot L$ is the set of all points $(x+r, y)$ where r is rational.

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[25] Loc. cit.

[26] See Footnote 23.